Quantum sensing of a quantum field

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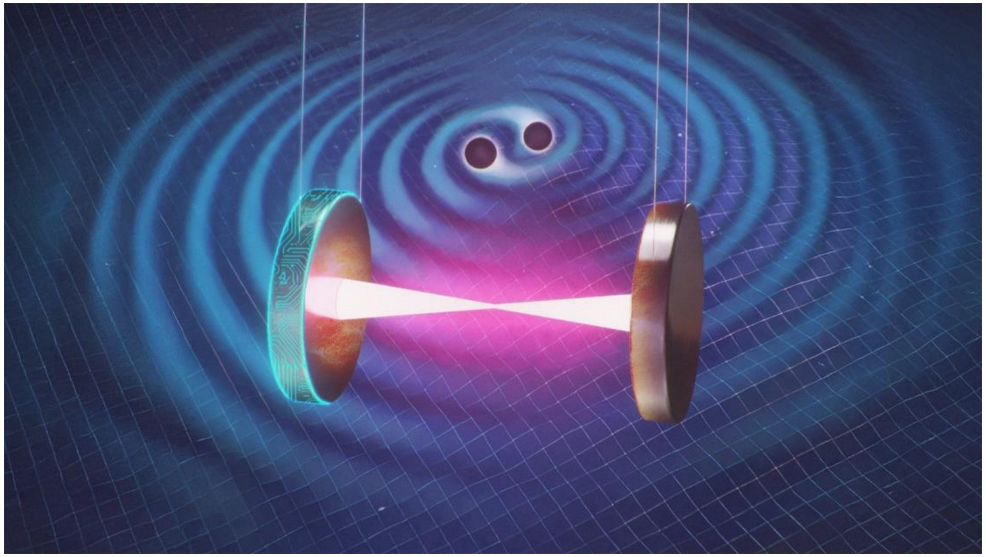
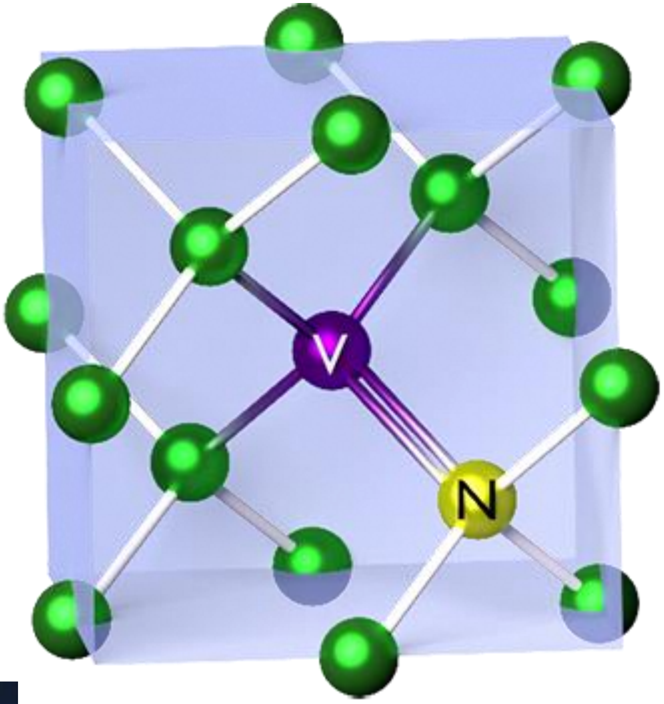
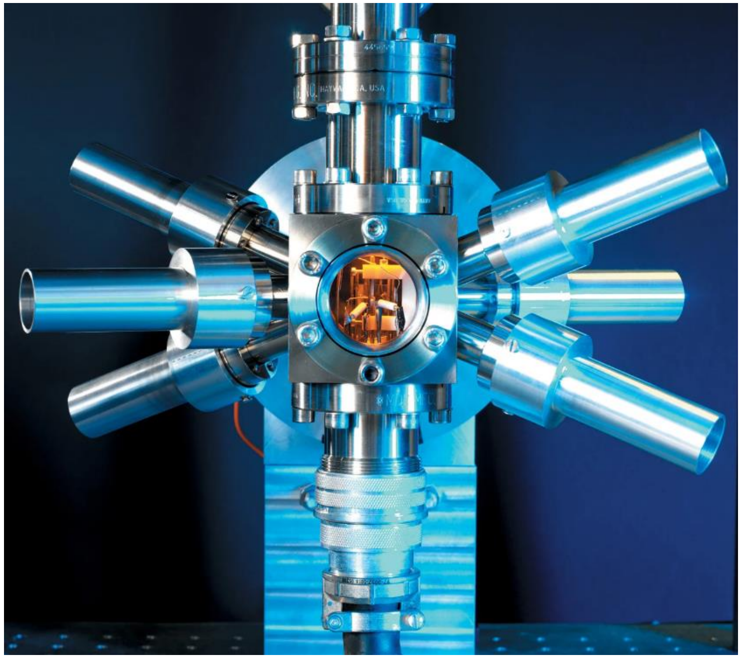
Motivation



Credit: Google DeepMind



Credit: NIST



Credit: Andrew Brookes/National Physical Laboratory/Science Source

Sensing

What is it?

We estimate a parameter θ by sampling a random variable X_θ, whose distribution p_θ depends on the parameter.

How?

We build a locally unbiased estimator from the probabilistic outcomes, and we denote the mean squared error of it as the precision of our estimation.

Sensing

There is a whole branch of statistics that deals with estimators…

NOT FOR US TODAY

Fisher information

- Fisher information as a ***benchmark*** of sensing.
- The precision of our estimation is proportional to the Fisher information.
- It quantifies the susceptibility of the prob. distributions on the parameter

$$\mathrm{FI}(X_\theta) = 8\left(\frac{1-\sqrt{\boldsymbol{p}_{\theta+\mathrm{d}\theta}}\sqrt{\boldsymbol{p}_\theta}}{\mathrm{d}\theta^2}\right).$$

QFI

- It is an optimization of the FI over all possible measurements that can be applied to the system (POVMs).
- The QFI is just the FI for the "optimal" measurement.

$$\begin{aligned}\mathrm{QFI}(\rho_\theta) &= \max_{\{E_x\}} \mathrm{FI}(X_\theta) \\ &= 8\left(\frac{1 - \left\|\sqrt{\rho_{\theta+\mathrm{d}\theta}}\sqrt{\rho_\theta}\right\|_1}{\mathrm{d}\theta^2}\right)\end{aligned}$$

QFI

It can also be computed as:

$$\mathrm{QFI}(\rho_\theta) = \mathrm{Tr}\left(L^2 \rho_\theta\right) \quad \text{for}$$

$$\frac{L\rho_\theta + \rho_\theta L}{2} = \frac{\mathrm{d}\rho_\theta}{\mathrm{d}\theta}$$

And for two-level systems it has a close formula

$$\mathrm{QFI}(\rho_\theta) = \frac{||\dot{\boldsymbol{r}}_\theta||^2 - ||\dot{\boldsymbol{r}}_\theta \times \boldsymbol{r}_\theta||^2}{1 - ||\boldsymbol{r}_\theta||^2}$$

Warning: for pure states we have to compute it differently.

Quantum sensing of a **quantum field**

Classical field

Semiclassical Rabi model:

$$H = E g \, \mathrm{i} \left(|e\rangle\langle g| - |g\rangle\langle e| \right)$$

Classical field

$$\mathrm{QFI}(\rho_{\tau|E}) = 4\tau^2 = 4g^2t^2.$$

1. Scales quadratically with the interaction time.
2. Independent of the field amplitude.

Quantum field

We consider the field to be in $|\alpha\rangle$ and we estimate α.

Crucial difference: the QFI of them is bounded!

$$\mathrm{QFI}\Big(\,|\alpha\rangle\langle\alpha|\,\Big) = 8\,\frac{1-|\langle\alpha|\alpha+\mathrm{d}\alpha\rangle|}{\mathrm{d}\alpha^2} = 4$$

The QFI cannot increase by post-processing so:

$$\mathrm{QFI}\Big(\rho_{\tau|\alpha}\Big) \leq 4$$

Single mode

Quantum field

Resonant Jaynes-Cummings model:

$$H = \mathrm{i}g\left(\sigma_+ a - \sigma_- a^\dagger\right)$$

↓

$$U_\tau = \begin{pmatrix} \cos\left(\tau\sqrt{\hat{n}}\right) & -\frac{\sin\left(\tau\sqrt{\hat{n}}\right)}{\sqrt{\hat{n}}}a^\dagger \\ \frac{\sin\left(\tau\sqrt{\hat{n}+1}\right)}{\sqrt{\hat{n}+1}}a & \cos\left(\tau\sqrt{\hat{n}+1}\right) \end{pmatrix}$$

Dynamical map

$$\mathcal{E}_{\tau|\alpha}[\bullet] = \sum_{i,j=0}^{3} G_{ij}^{\alpha} L_i \bullet L_j^{\dagger}$$

$$\{L_0, L_1, L_2, L_3\} = \{|g\rangle\langle g|\,, |e\rangle\langle e|\,, \sigma_-, \sigma_+\}$$

$$G_{ij}^{\alpha} = \langle\alpha|\, f_j(\hat{n}) f_i(\hat{n})\, |\alpha\rangle = \sum_{n=0}^{\infty} \mathrm{P}(n|\alpha) f_i(n) f_j(n) = \mathbb{E}[f_i(\hat{n}) f_j(\hat{n})]$$

where

$$f_0(\hat{n}) = \cos(\tau\sqrt{\hat{n}}) \qquad f_2(\hat{n}) = -\sin(\tau\sqrt{\hat{n}})\frac{\sqrt{\hat{n}}}{\alpha}$$

$$f_1(\hat{n}) = \cos(\tau\sqrt{\hat{n}+1}) \qquad f_3(\hat{n}) = \sin(\tau\sqrt{\hat{n}+1})\frac{\alpha}{\sqrt{\hat{n}+1}}.$$

Dynamical map

For the ground state:

$$x_\alpha^{(g)} = 2\,\mathbb{E}\left[\frac{\alpha}{\sqrt{\hat{n}+1}}\cos(\tau\sqrt{\hat{n}})\sin(\tau\sqrt{\hat{n}+1})\right] \qquad z_\alpha^{(g)} = 2\,\mathbb{E}\left[\cos^2(\tau\sqrt{\hat{n}})\right]-1 \tag{1}$$

$$\dot{x}_\alpha^{(g)} = 2\,\mathbb{E}\left[\frac{(2\hat{n}-2\alpha^2+1)}{\sqrt{\hat{n}+1}}\cos(\tau\sqrt{\hat{n}})\sin(\tau\sqrt{\hat{n}+1})\right] \qquad \dot{z}_\alpha^{(g)} = 2\,\mathbb{E}\left[\frac{2\hat{n}-2\alpha^2}{\alpha}\cos^2(\tau\sqrt{\hat{n}})\right]$$

Only analytically solvable for $\alpha \ll 1, \alpha \gg 1$

At vacuum $\alpha = 0$ → Rabi oscillations:

$$\mathcal{E}_{\tau|0}[\bullet] = K_0 \bullet K_0^\dagger + K_1 \bullet K_1^\dagger \quad \text{with}$$

$$K_0 = \begin{pmatrix} 0 & \sin(\tau) \\ 0 & 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \cos(\tau) \end{pmatrix}$$

$\alpha \ll 1$

For $0 < \alpha \ll 1$, we use

$$|\alpha\rangle \propto |0\rangle + \alpha\,|1\rangle + \frac{\alpha^2}{2}\,|2\rangle$$

We obtain

$$\rho^{(g)}_{\tau|\alpha} = \begin{pmatrix} 1 - \mathrm{s}_\tau^2\alpha^2 & \alpha\,\mathrm{s}_\tau \\ \alpha\,\mathrm{s}_\tau & \alpha^2\mathrm{s}_\tau^2 \end{pmatrix},$$

and $\mathrm{QFI}[\rho^{(g)}_{\tau|\alpha}] = 4\sin^2(\tau) + O(\alpha^2)$. It saturates the upper bound!

JC is optimal for low α!!

$\alpha \gg 1$

Crude approximation: $a^\dagger |\alpha\rangle \approx \alpha^* |\alpha\rangle$

only valid for $\tau \ll 1$

We can solve the model analytically though!

$\alpha \gg 1$

Remember: $G_{ij} = \sum_{n=0}^{\infty} \mathrm{P}(n|\alpha) f_i(n) f_j(n) = \mathbb{E}[f_i(\hat{n}) f_j(\hat{n})]$

where $f_i(n) f_j(n) = \Gamma_{ij}(n) \sum_{k,\pm} c_{i,j}(k,\pm)\, e^{\pm \mathrm{i} S_k(n)}$,
$\Gamma_{ij}(n) \in \{1, \frac{\sqrt{n}}{\alpha}, \frac{\alpha}{\sqrt{n+1}}, \frac{\sqrt{n}}{\sqrt{n+1}}\}$,
and

$$\left(S_0(n), S_1(n), S_2(n), S_3(n)\right) = \left(2\tau\sqrt{n}, 2\tau\sqrt{n+1}, \tau(\sqrt{n}+\sqrt{n+1}), \tau(\sqrt{n}-\sqrt{n+1})\right)$$

$\alpha \gg 1$

First, we expand around $\hat{\delta} = \frac{\hat{n}-\alpha^2}{\alpha} = O(1)$
and get $\Gamma_{ij}(n) = O(\alpha^{-1})$.
So, we "only" need to calculate terms as

$$\sum_{n=0}^{\infty} P(n|\alpha)\Gamma_{ij}(n)e^{\mathrm{i}S_k(n)} = \sum_{n=0}^{\infty} P(n|\alpha)e^{\mathrm{i}S_k(n)} + O(\alpha^{-1}).$$

$\alpha \gg 1$

There are two diferent regimes w.r.t. the phases $S_i(n)$

1. Linear regime: $\tau = O(1)$
2. Rapid regime: $\tau \geq O(\alpha)$

$\alpha \gg 1$, lin. regime

Expanding around $\hat{\delta} = \frac{\hat{n}-\alpha^2}{\alpha} = O(1)$

we have

$$S_0(n), S_1(n), S_2(n) = 2\alpha\tau + \tau\frac{n-\alpha^2}{\alpha} + \tau O(\alpha^{-1})$$

$$S_3(n) = -\frac{\tau}{2\alpha} + \frac{\tau}{4\alpha^2}\frac{n-\alpha^2}{\alpha} + \tau O(\alpha^{-3}).$$

There are no more square roots, so we can use the moment generating function of the Poissonian

$$\mathbb{E}[e^{i\mu n}] = \exp(-\alpha^2 + \alpha^2 e^{i\mu})$$

$\alpha \gg 1$, lin. regime

Voilà!

$$\mathcal{I}_{k,0} := \sum_{n=0}^{\infty} P(n|\alpha) e^{\mathrm{i} S_k(n)} = \exp\left(-\frac{\tau^2}{2} + \mathrm{i}\, 2\alpha\, \tau\right) \qquad \text{for } \; k = 1, 2, 3.$$

$$\mathcal{I}_{3,0} := \sum_{n=0}^{\infty} P(n|\alpha) e^{\mathrm{i} S_3(n)} = \exp\left(-\frac{\tau^2}{32\alpha^4} - \mathrm{i}\, \frac{\tau}{2\alpha}\right)$$

Note: these formulas are only valid up to $O\left(\frac{\tau}{\alpha}\right)$ and $O\left(\frac{\tau}{\alpha^3}\right)$ respectively.

$\alpha \gg 1$, rap. regime *

The phases are no longer linear in n. We now have to use the Poissonian summation formula

$$\sum_{n=0}^{\infty} P(n|\alpha)e^{\mathrm{i}S_k(n)} = \xcancel{\frac{1}{2}P(0|\alpha)e^{\mathrm{i}S_k(0)}} + \sum_{\nu=-\infty}^{\infty} \int_0^{\infty} P(n|\alpha)e^{\mathrm{i}S_k(n)-\mathrm{i}2\pi n\nu}\mathrm{d}n$$

$\alpha \gg 1$, rap. regime

Key points to solve the integrals:

- The envelope (Poiss. distr) varies more slowly than the exponent.
- We compute the maximum of the exponent and then use the stationary phase approximation.
- We approximate the square root of a Poissonian random variable as a Gaussian.
- ν represents the different revivals.

$\alpha \gg 1$, whole ev.

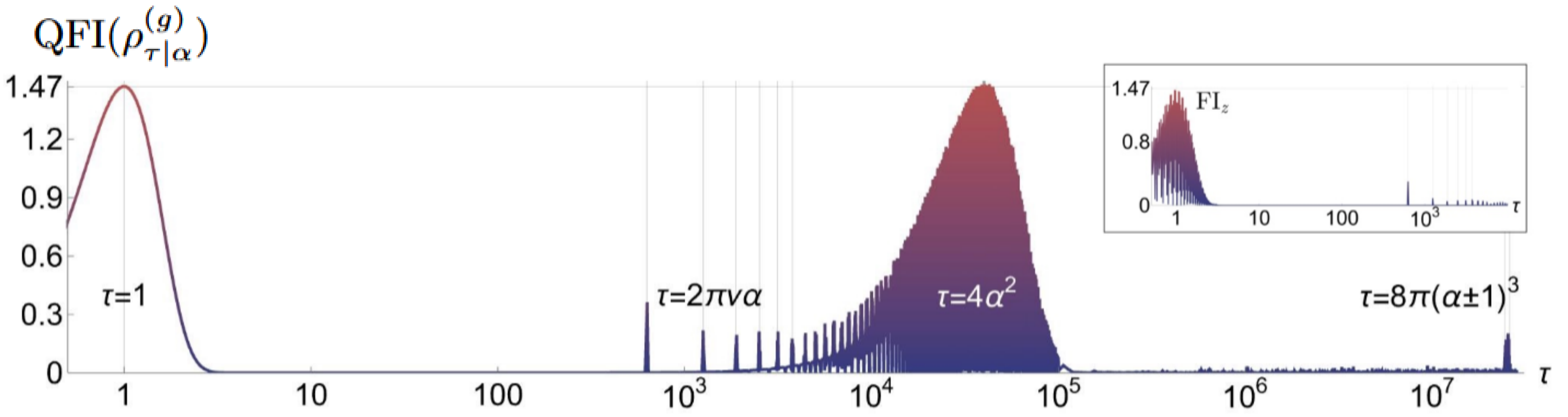
$$G = \frac{1}{2}\mathbb{1} + \sum_{\nu=0}^{\infty} (\mathcal{G}_\nu^{(0)} + \mathcal{G}_\nu^{(3)}) \qquad \text{with}$$

$$\mathcal{G}_\nu^{(0)} = \frac{1}{2}\begin{pmatrix} \mathcal{I}_{0,\nu}^{\mathrm{Re}} & (-1)^\nu \mathcal{I}_{0,\nu}^{\mathrm{Re}} & -\mathcal{I}_{0,\nu}^{\mathrm{Im}} & (-1)^\nu \mathcal{I}_{0,\nu}^{\mathrm{Im}} \\ (-1)^\nu \mathcal{I}_{0,\nu}^{\mathrm{Re}} & \mathcal{I}_{0,\nu}^{\mathrm{Re}} & -(-1)^\nu \mathcal{I}_{0,\nu}^{\mathrm{Im}} & \mathcal{I}_{0,\nu}^{\mathrm{Im}} \\ -\mathcal{I}_{0,\nu}^{\mathrm{Im}} & -(-1)^\nu \mathcal{I}_{0,\nu}^{\mathrm{Im}} & -\mathcal{I}_{0,\nu}^{\mathrm{Re}} & (-1)^\nu \mathcal{I}_{0,\nu}^{\mathrm{Re}} \\ (-1)^\nu \mathcal{I}_{0,\nu}^{\mathrm{Im}} & \mathcal{I}_{0,\nu}^{\mathrm{Im}} & (-1)^\nu \mathcal{I}_{0,\nu}^{\mathrm{Re}} & -\mathcal{I}_{0,\nu}^{\mathrm{Re}} \end{pmatrix} \qquad \mathcal{G}_\nu^{(3)} = \frac{1}{2}\begin{pmatrix} & \mathcal{I}_{3,\nu}^{\mathrm{Re}} & & -\mathcal{I}_{3,\nu}^{\mathrm{Im}} \\ \mathcal{I}_{3,\nu}^{\mathrm{Re}} & & -\mathcal{I}_{3,\nu}^{\mathrm{Im}} & \\ & -\mathcal{I}_{3,\nu}^{\mathrm{Im}} & & -\mathcal{I}_{3,\nu}^{\mathrm{Re}} \\ -\mathcal{I}_{3,\nu}^{\mathrm{Im}} & & -\mathcal{I}_{3,\nu}^{\mathrm{Re}} & \end{pmatrix}$$

$$\mathcal{I}_{0,\nu} = \begin{cases} \exp\left(-\frac{\tau^2}{2} + \mathrm{i}\,2\alpha\,\tau\right) & \nu = 0 \\ \frac{1}{\sqrt{\pi\nu}} \exp\left(-\frac{(\tau - 2\pi\nu\alpha)^2}{2\pi^2\nu^2}\right) \exp\left(\mathrm{i}\left(\frac{\tau^2}{2\pi\nu} - \frac{\pi}{4}\right)\right) & \nu \geq 1 \end{cases}$$

$$\mathcal{I}_{3,\nu} = \begin{cases} \exp\left(-\frac{\tau^2}{32\alpha^4} - \mathrm{i}\,\frac{\tau}{2\alpha}\right) & \nu = 0 \\ \frac{1}{\sqrt{3\pi\nu}} \exp\left(-\frac{\left(\tau^{1/3} - 2(\pi\nu)^{1/3}\alpha\right)^2}{2(\pi\nu)^{2/3}}\right)(-1)^\nu \exp\left(-\mathrm{i}\frac{3(\pi\nu)^{1/3}\tau^{2/3}}{2} - \mathrm{i}\frac{\pi}{4}\right) & \nu \geq 1 \end{cases}.$$

$\alpha \gg 1$


$\mathrm{QFI}(\rho^{(g)}_{\tau|\alpha})$
1.47
1.2
0.9
0.6
0.3
0
1
10
100
10^3
10^4
10^5
10^6
10^7
τ
τ=1
$\tau=2\pi\nu\alpha$
$\tau=4\alpha^2$
$\tau=8\pi(\alpha\pm1)^3$
FI_z
0.8

Numerical results

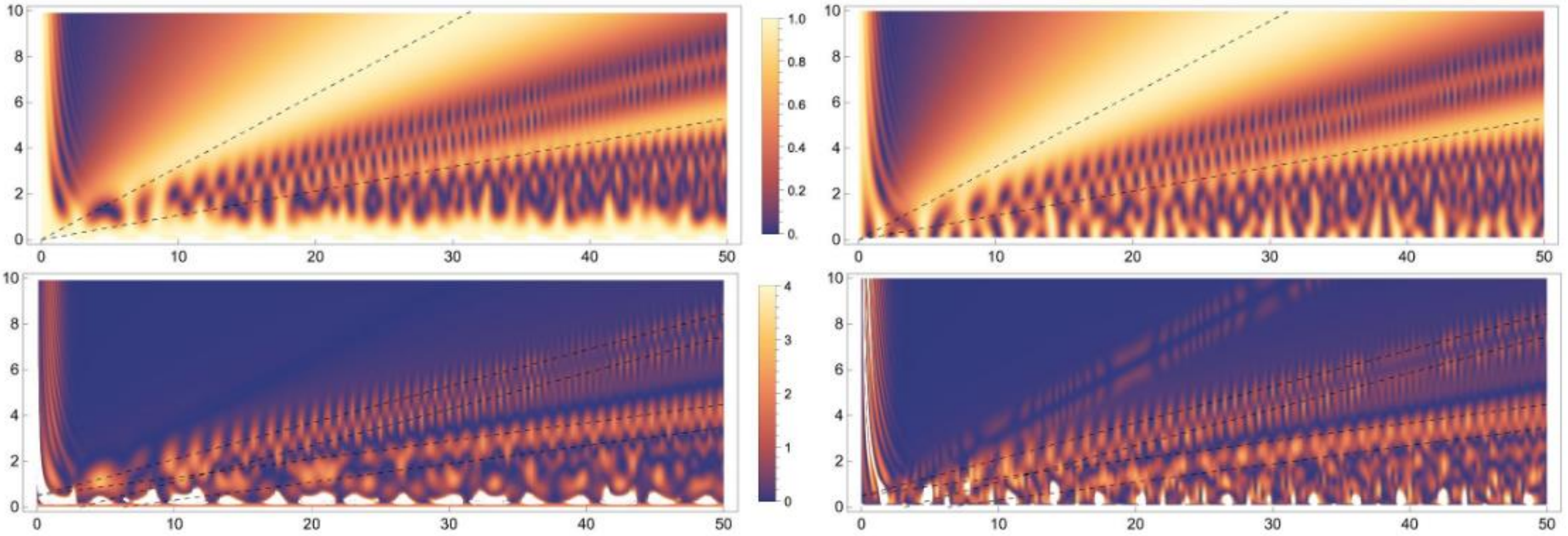


FIG. 2. From top to bottom: the purity $\mathcal{P}[\rho_{\tau|\alpha}^{(g/e)}]$ and the quantum Fisher information $\mathrm{QFI}[\rho_{\tau|\alpha}^{(g/e)}]$ of the atomic state as function of time τ (x-axis) and coherent state amplitude α (y-axis). The left column corresponds to the ground state, and the right to the excited. The guidelines represent the different "revival" times derived analytically in the asymptotic limit (see Appendix D). $1 - \mathcal{P}[\rho_{\tau|\alpha}^{(g/e)}]$ measures the atom-field entanglement.

Numerical results

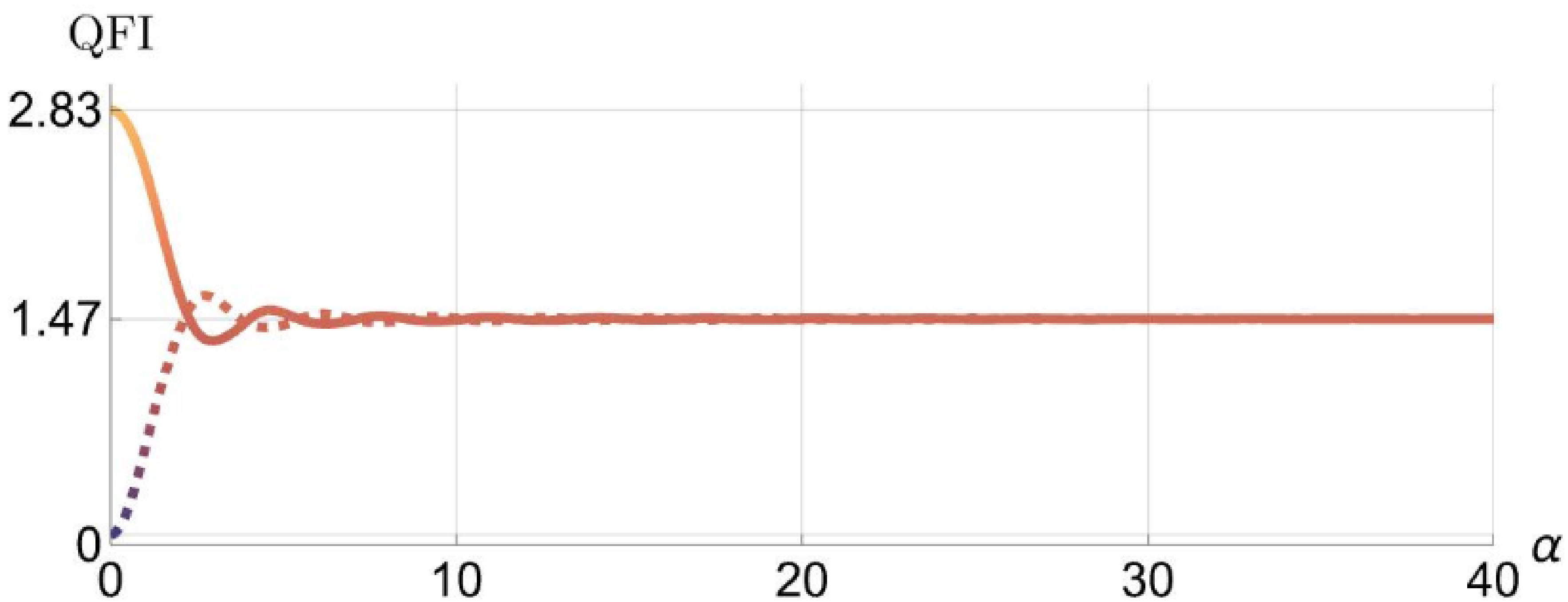


FIG. 3. The $\mathrm{QFI}[\rho^{(g/e)}_{\tau=1|\alpha}]$ of the ground (full line) and excited (dashed line) initial state after an interaction with the coherent mode of duration $\tau = 1$, as a function of α. We see that it converges relatively fast to the asymptotic value $4/e \approx 1.47$.

It seems that $\tau \approx 1$, it is very optimal!

Multiple modes

1. Sequence of N interactions
2. Infinite modes limit
3. Continuous field limit

N interactions

$$\mathcal{E}_{\tau|\alpha}^{\circ N} = \underbrace{\mathcal{E}_{\tau|\alpha} \circ \cdots \circ \mathcal{E}_{\tau|\alpha}}_{N}$$

The theoretical bound is now:

$$\mathrm{QFI}[\rho_{\tau,N|\alpha}] \leq 4N$$

N interactions

For $\alpha \gg 1$, we see that it is only beneficial to apply the map more than one time when $\tau \ll 1$. We have:

$$\mathcal{E}^{\circ N}_{\tau|\alpha} := \begin{pmatrix} \mathbb{1} \\ Q_0^N \left(\cos(N\Phi_0)\sigma_x - \sin(N\Phi_0)\sigma_z \right) \\ \sigma_y \\ Q_0^N \left(\sin(N\Phi_0)\sigma_x + \cos(N\Phi_0)\sigma_z \right) \end{pmatrix}$$

with $Q_0^N = e^{-\tau^2/2}$ and $\Phi_0 = 2\alpha\tau$. So

$$\mathrm{QFI} = 4N^2\tau^2 e^{-N\tau^2} \longrightarrow \max_{\tau} \mathrm{QFI}[\rho_{\tau,N|\alpha}] = \frac{4}{e}N \approx 1.47N$$

N interactions

If $T = N\tau$, the theoretical bound reads (for any fixed τ)

$$\mathrm{QFI}[\rho_{\tau,N|\alpha}] \leq 4\frac{T}{\tau}$$

What happens at $\tau \to 0$??

Infinite modes limit

If $\tau \to 0$, and $N = \frac{1}{dt} \to \infty$, and $\alpha = cst$, the evolution is simply

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{t|\alpha} = -\mathrm{i}g\,\alpha\,[\sigma_y, \rho_{t|\alpha}]$$

which is the semiclassical Rabi model!!

$$\mathrm{QFI} = 4g^2t^2$$

Infinite modes limit

It is <u>non-physical</u>

1. Diverging energy of the field $N\alpha^2 \to \infty$
2. Diverging photon flux $\frac{\alpha^2}{dt} \to \infty$

To circumvent this, we keep these quantitites bounded and go to the cont. limit.

Cont. field limit

We now define $\tau = \sqrt{\kappa\, dt}$, and $\alpha^2 = \epsilon^2 dt$, where now ϵ is the quantity to estimate.

So:

$$H(t) = \mathrm{i}\sqrt{\kappa}\Big(\sigma_-(\varepsilon + a^\dagger(t)) - \sigma_+(\varepsilon + a(t))\Big)$$

$$s = \kappa t \qquad \downarrow \qquad \bar{\varepsilon} = \frac{\varepsilon}{\sqrt{\kappa}}$$

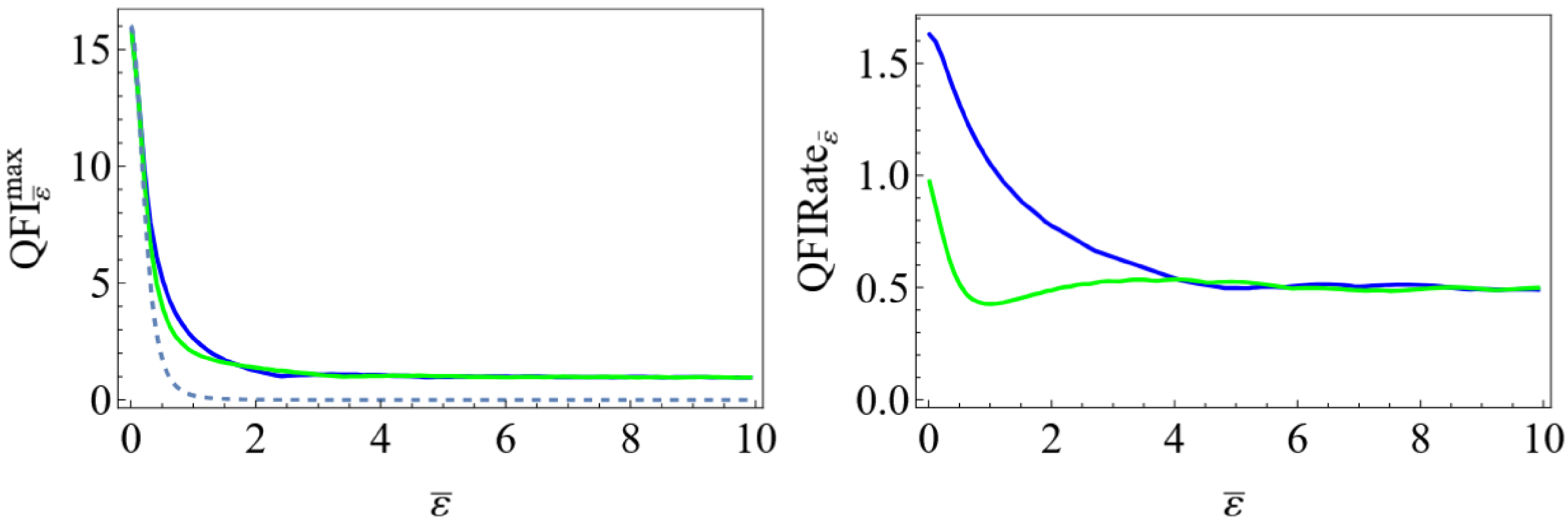
$$\frac{\mathrm{d}}{\mathrm{d}s}\rho_{s|\bar{\varepsilon}} = -\mathrm{i}\,\bar{\varepsilon}\,[\sigma_y, \rho_{s|\bar{\varepsilon}}] + \mathcal{L}_{\sigma_-}[\rho_{s|\bar{\varepsilon}}]$$

Cont. field limit

Theoretical upper bound on $\mathrm{QFIRate}_{\bar{\varepsilon}} := \sup_s \frac{\mathrm{QFI}[\rho_{s|\bar{\varepsilon}}]}{s}$

$$\mathrm{QFIRate}_{\bar{\varepsilon}} \leq 4$$

Cont. field limit



$$\mathrm{QFI}^{\max}_{\bar{\varepsilon}} = \sup_s \mathrm{QFI}[\rho_{s|\bar{\varepsilon}}] \quad \mathrm{QFIRate}_{\bar{\varepsilon}} := \sup_s \frac{\mathrm{QFI}[\rho_{s|\bar{\varepsilon}}]}{s}$$

blue → ground state
green → excited state
dashed line → the steady QFI: $\mathrm{QFI}[\rho^*_{\bar{\varepsilon}}] = \left(\frac{4}{(1+8\,\bar{\varepsilon}^2)}\right)^2$

Outlook

Conclusions:

- The non-accuracy of the semiclassical model.
- The importance of the JC.
- Possibility to quantify the optimality of a given interaction.

Outlook

Open questions:

- Use other states of light, e.g., squeezed states.
- Is there an interaction that is always optimal regardless of α?
- Bayesian framework?
- More difficult dynamics?

THANK YOU

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