

FROM CHANDRASEKHAR'S SOLUTION,
THROUGH DAMSKI-CAHNDRASEKHAR 'S
SHOCKWAVES, THROUGH A $SO(2, 1)$ MAP,
ON TO THE TRIANGULAR
GROSS-PITAEVSKII BREATHERS:
REINTERPRETING
THE SHI-GAO-ZHAI FERMI-BOSE
MAP SOLUTION

MAXIM OLSHANII, D. DESHOMMES, J. TORRENTS, M.
GONCHENKO, V. DUNJKO, G. E. ASTRAKHARCHIK



PUC
JUN 17
2021



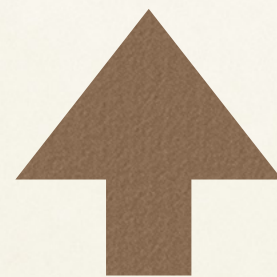
Overview

Trapped triangular GPE breathers
[Saint-Jalm, Dalibard, et al (2019)]

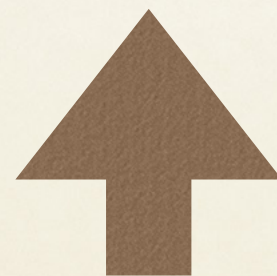
Expalnation for triang. breathers:
interacting bosons $\leftrightarrow$ free fermions
map; simplexes only, scale-
invariant interactions only, both
trapped [Shi-Gao-Zhai (2020)]



$SO(2, 1)$ map to untrapped particles [Pitaevskii-Rosch (1997)]



Map to a Nonlinear Transport Equation [Damski (2004)]



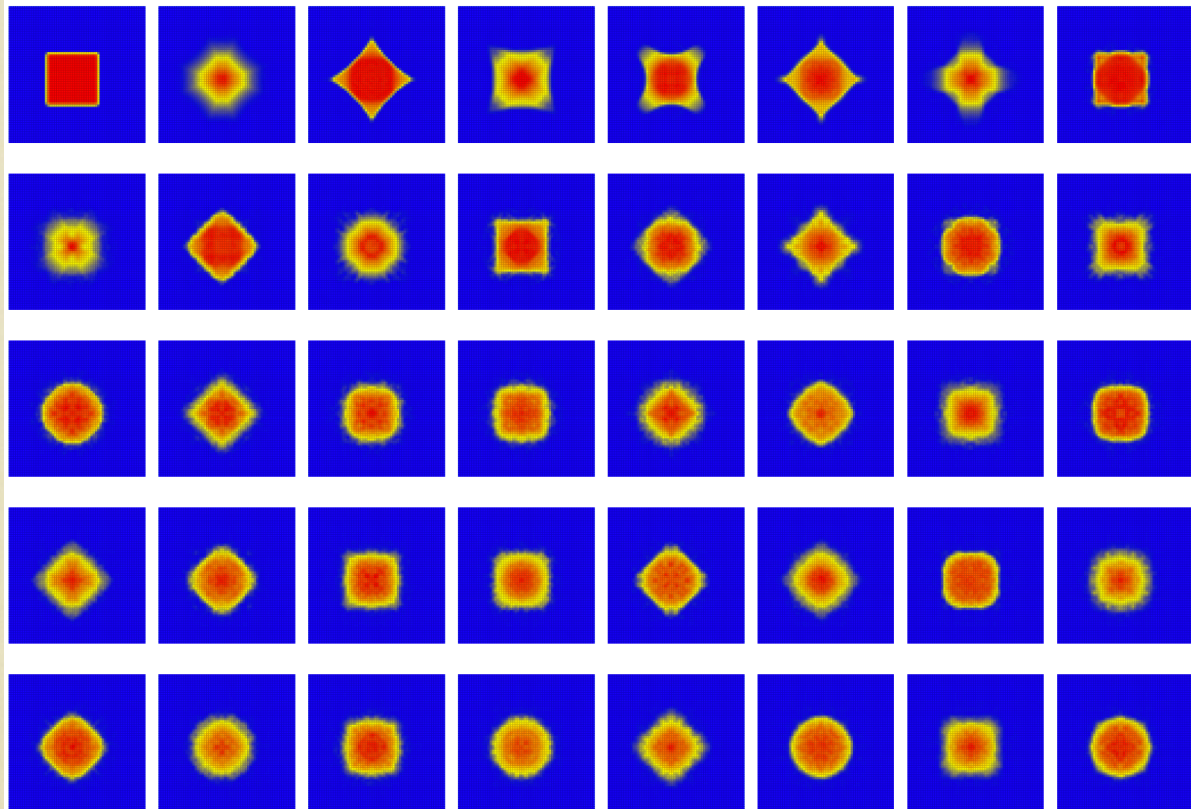
Chandrasekhar's solution of the Nonlinear Transport
Equation [Chandrasekhar (1943)]



The phenomenon: triangular GPE breathers: the ENS experiment

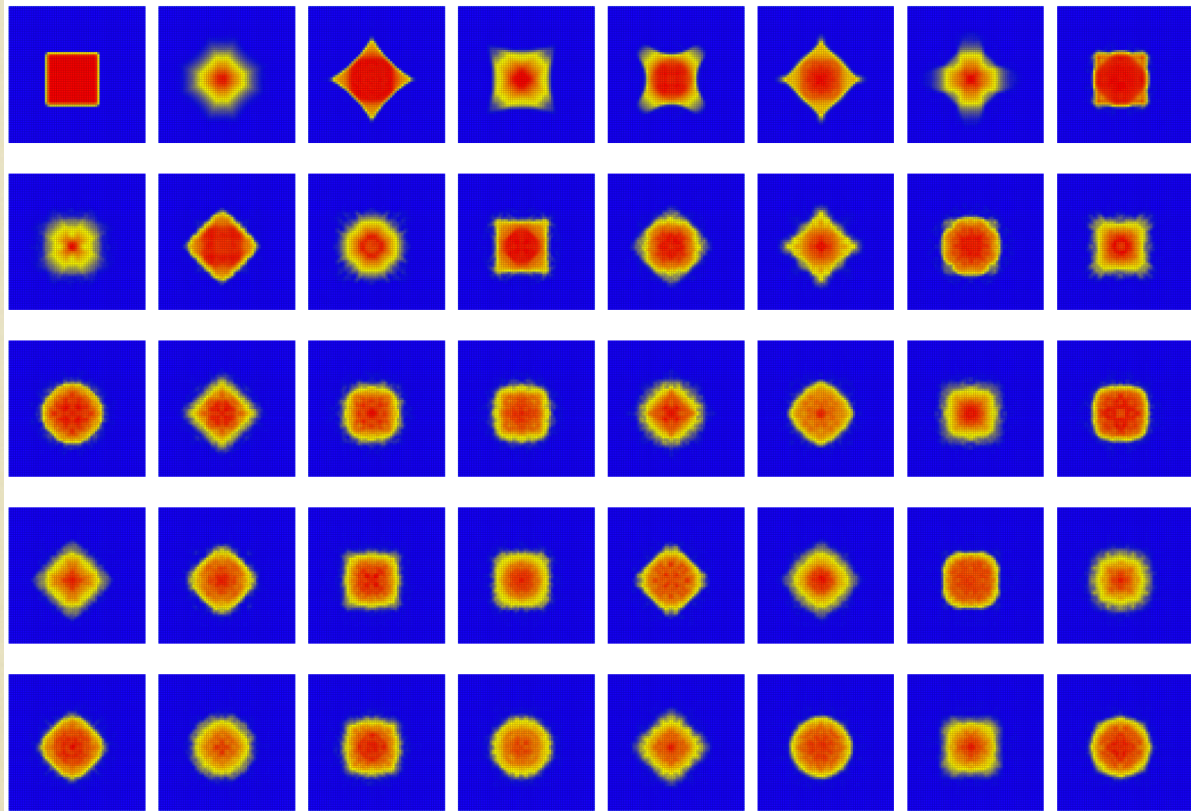
R. Saint-Jalm, P. Castilho, E. L. Cerf, B. Bakkali-Hassani, J.-L. Ville, S. Nascimbene, J. Beugnon and J. Dalibard, Dynamical symmetry and breathers in a two-dimensional Bose gas, Phys. Rev. X 9, 021035 (2019).

BEC released from a flat-bottom polygonal corral to a harmonic trap



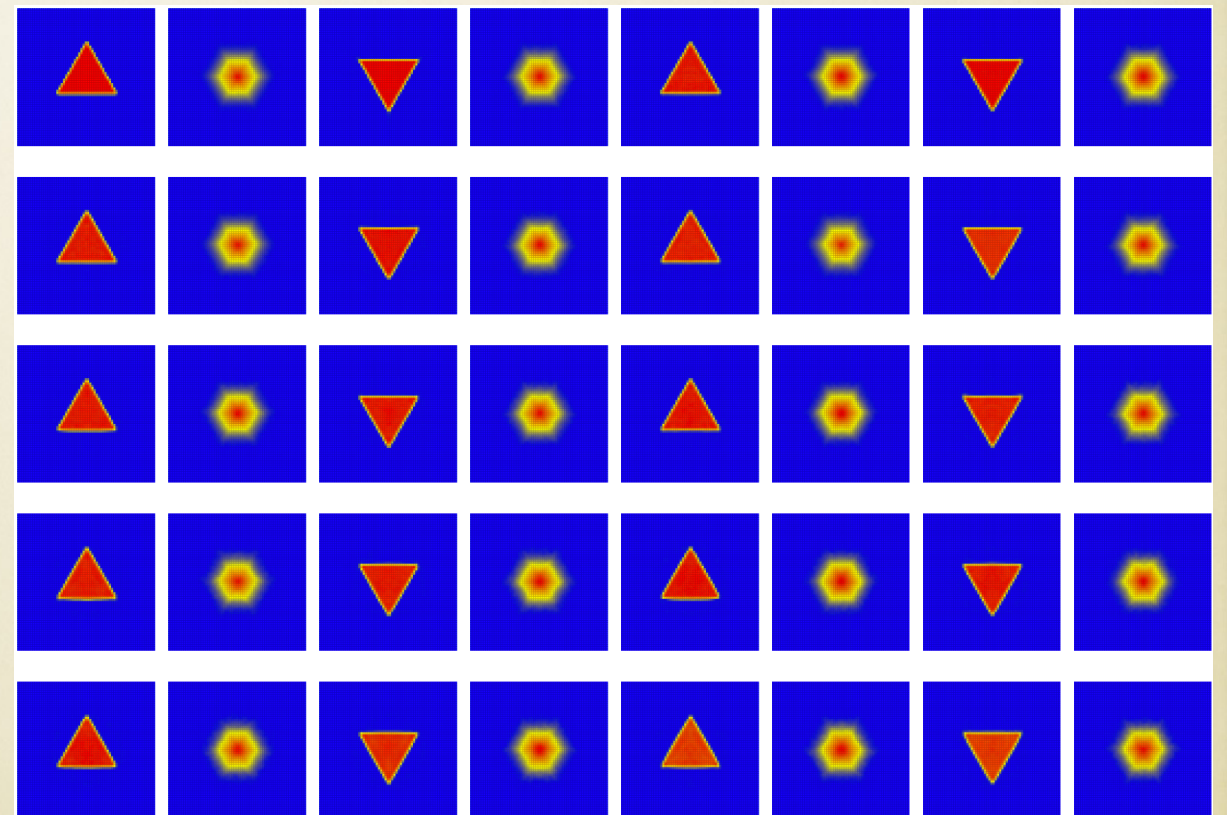
squares and others thermalize

BEC released from a flat-bottom polygonal corral to a harmonic trap

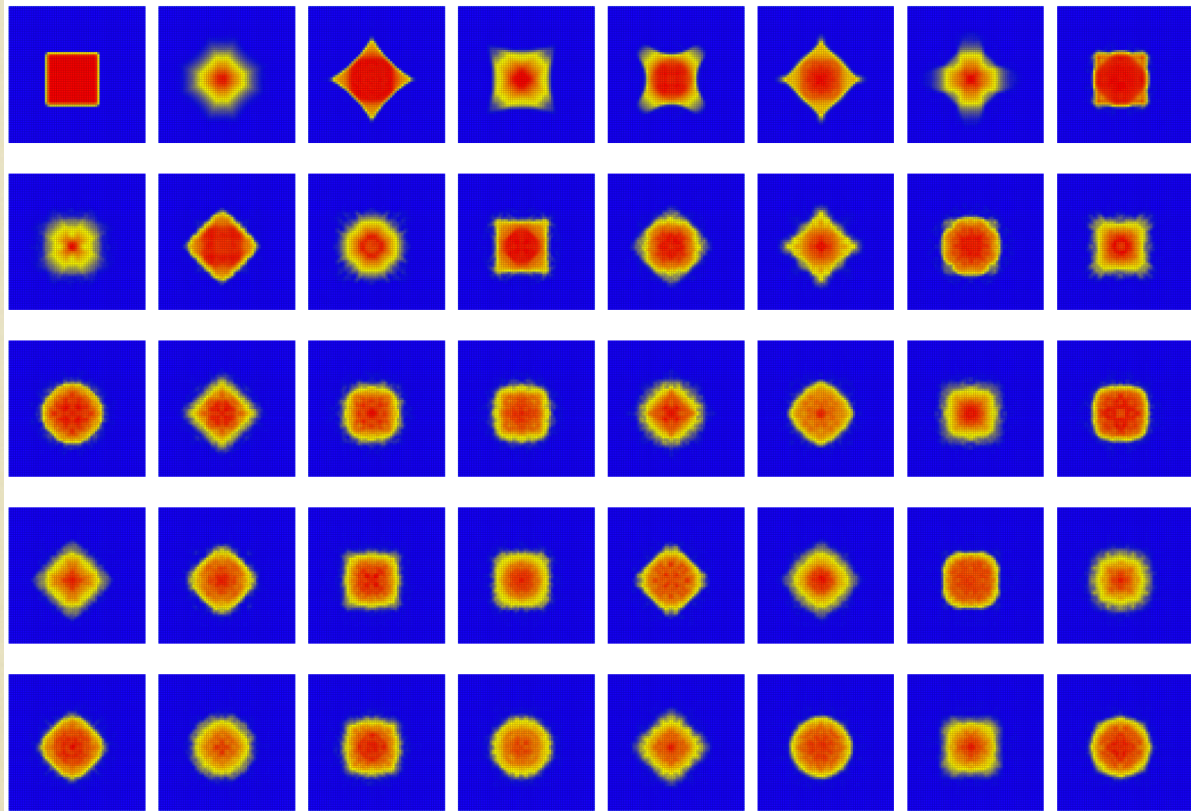


squares and others thermalize

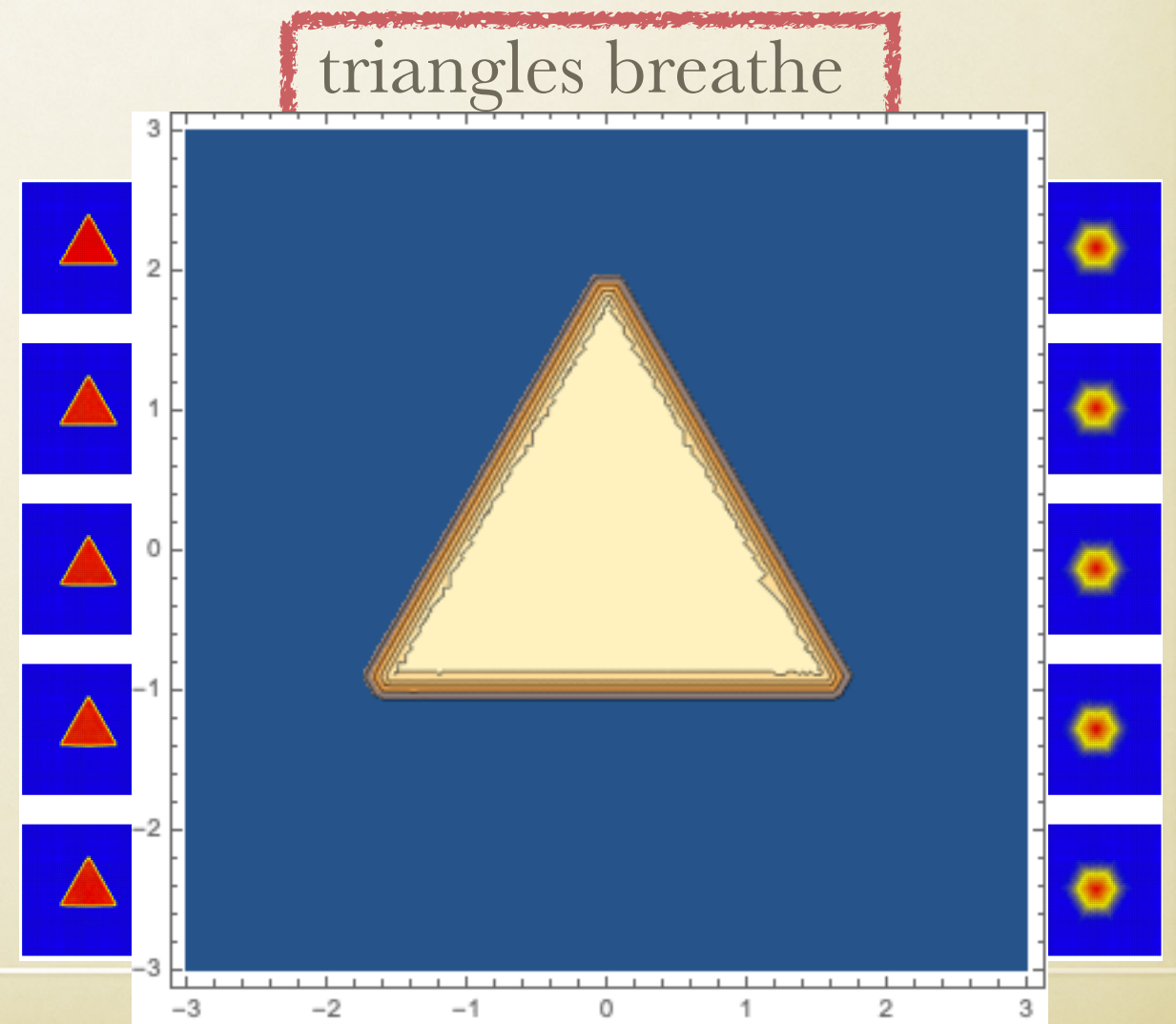
triangles breathe



BEC released from a flat-bottom polygonal corral to a harmonic trap



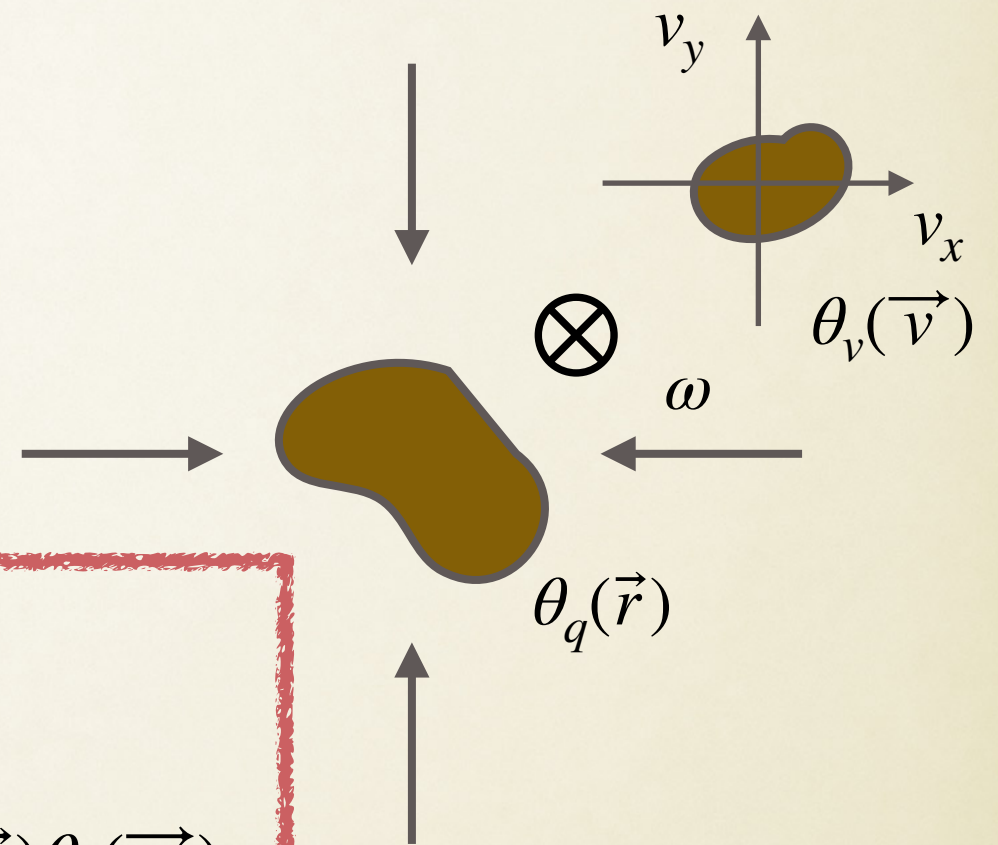
squares and others thermalize



Explanation: Shi-Gao-Zhai
solution: a novel
Fermi-Bose map

Z.-Y. Shi, C. Gao and H. Zhai, Idealized hydrodynamics (2020), arXiv/2011.01415.

Free 2D fermions first



Question:

Assume:

$$\rho(\vec{r}, \vec{v}, t = 0) = \left(\frac{m}{2\pi\hbar} \right)^{d=2} \theta_q(\vec{r}) \theta_v(\vec{v})$$

Under what circumstances is

$$\frac{\partial}{\partial t} \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{m} \vec{\nabla} \mu(n) - \omega^2 \vec{r}$$

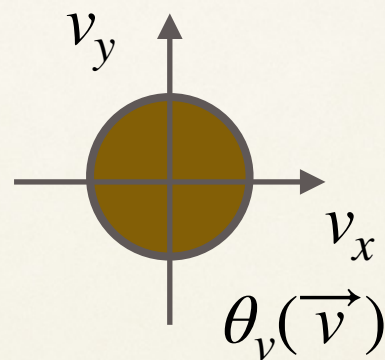
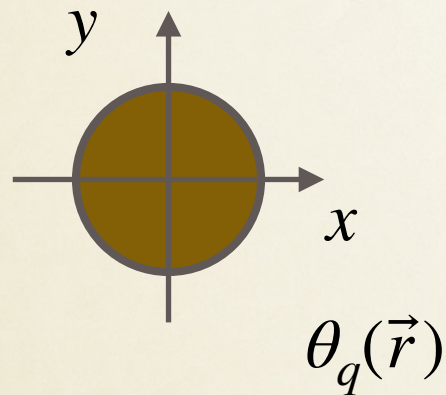
exact, for some $\mu(n)$?

$$n(\vec{r}, t) \equiv \int d^2v \rho(\vec{r}, \vec{v}, t)$$

$$\vec{v}(\vec{r}, t) \equiv n(\vec{r}, t)^{-1} \int d^2v \vec{v} \rho(\vec{r}, \vec{v}, t)$$

Free 2D fermions first

Wrong answer:

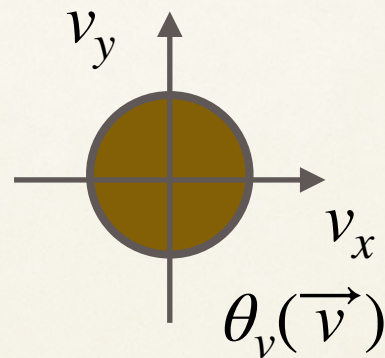
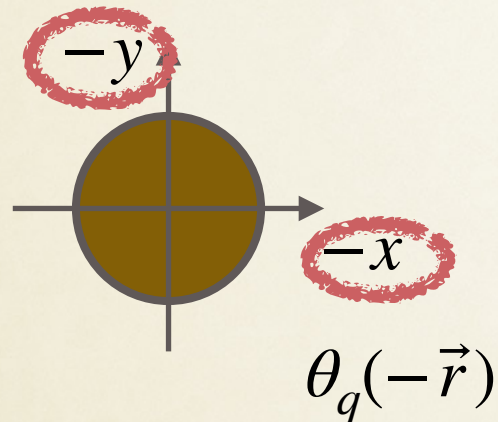


$$\rho(\vec{r}, \vec{v}, t) = \left(\frac{m}{2\pi\hbar} \right)^2 \theta_q(\vec{r} \cos(\omega t) - \vec{v} \sin(\omega t)/\omega) \theta_v(\omega \vec{r} \sin(\omega t) + \vec{v} \cos(\omega t))$$

$$w(\vec{v} | \vec{r}, t) =$$

Free 2D fermions first

Wrong answer:

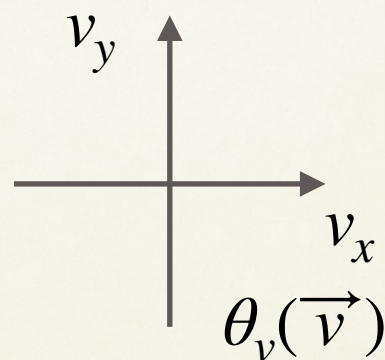
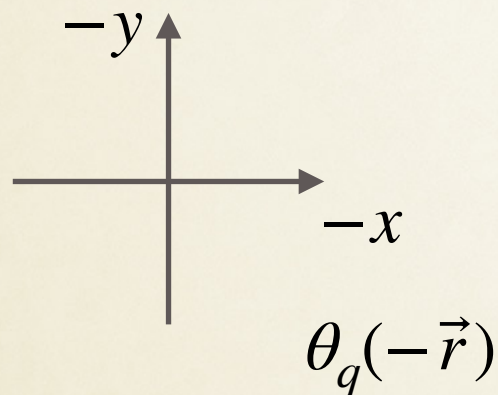


$$\rho(\vec{r}, \vec{v}, t) = \left(\frac{m}{2\pi\hbar} \right)^2 \theta_q(\vec{r} \cos(\omega t) - \vec{v} \sin(\omega t)/\omega) \theta_v(\omega \vec{r} \sin(\omega t) + \vec{v} \cos(\omega t))$$

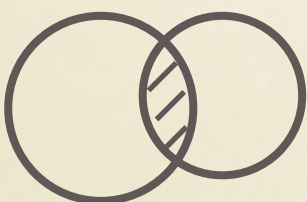
$$w(\vec{v} | \vec{r}, t) =$$

Free 2D fermions first

Wrong answer:

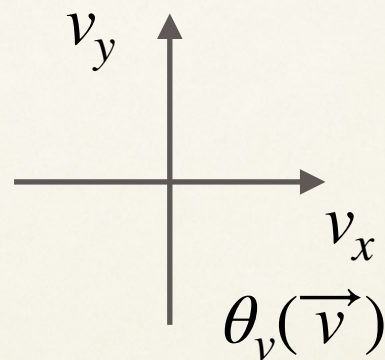
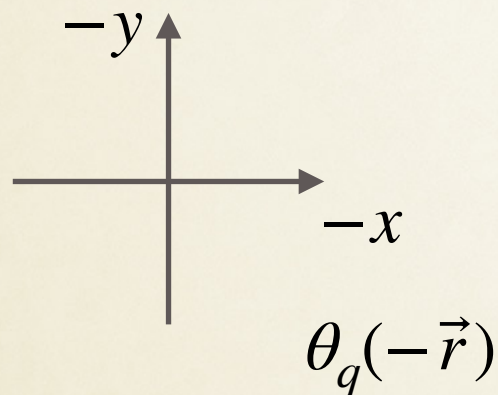


$$\rho(\vec{r}, \vec{v}, t) = \left(\frac{m}{2\pi\hbar} \right)^2 \theta_q(\vec{r} \cos(\omega t) - \vec{v} \sin(\omega t)/\omega) \theta_v(\omega \vec{r} \sin(\omega t) + \vec{v} \cos(\omega t))$$

$$w(\vec{v} | \vec{r}, t) =$$


Free 2D fermions first

Wrong answer:



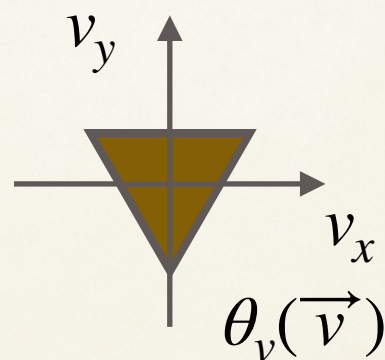
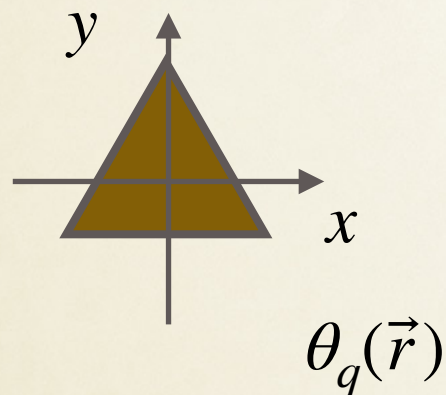
$$\rho(\vec{r}, \vec{v}, t) = \left(\frac{m}{2\pi\hbar} \right)^2 \theta_q(\vec{r} \cos(\omega t) - \vec{v} \sin(\omega t)/\omega) \theta_v(\omega \vec{r} \sin(\omega t) + \vec{v} \cos(\omega t))$$

$$w(\vec{v} | \vec{r}, t) = \text{Venn diagram with two overlapping circles, the intersection shaded} \neq \text{Venn diagram with one shaded circle} \Rightarrow$$

$$\Rightarrow \frac{\int d^2v (\vec{v} - \bar{\vec{v}})^2 \rho(\vec{r}, \vec{v}, t)}{\int d^2v \rho(\vec{r}, \vec{v}, t)} \neq \text{const} \Rightarrow \epsilon \neq \epsilon(n) \Rightarrow \nexists \mu(n)$$

Free 2D fermions first

Correct answer:

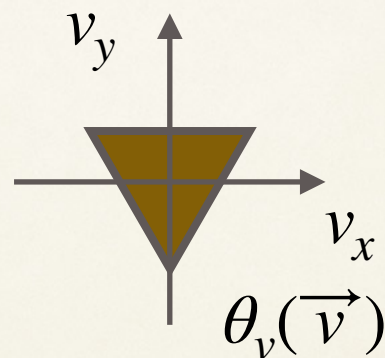
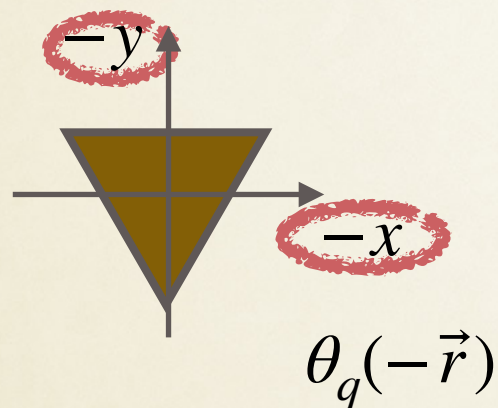


$$\rho(\vec{r}, \vec{v}, t) = \left(\frac{m}{2\pi\hbar} \right)^2 \theta_q(\vec{r} \cos(\omega t) - \vec{v} \sin(\omega t)/\omega) \theta_v(\omega \vec{r} \sin(\omega t) + \vec{v} \cos(\omega t))$$

$$w(\vec{v} | \vec{r}, t) =$$

Free 2D fermions first

Correct answer:

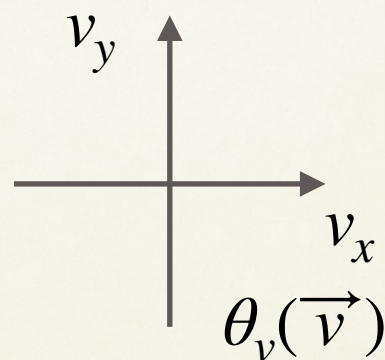
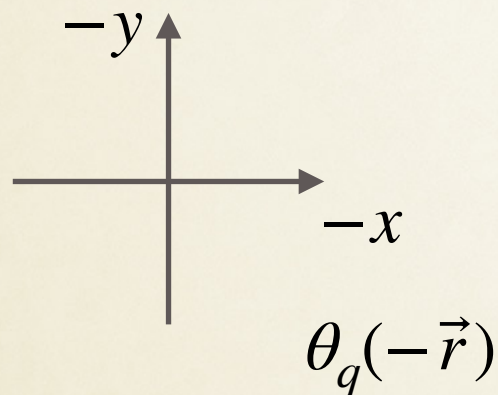


$$\rho(\vec{r}, \vec{v}, t) = \left(\frac{m}{2\pi\hbar} \right)^2 \theta_q(\vec{r} \cos(\omega t) - \vec{v} \sin(\omega t)/\omega) \theta_v(\omega \vec{r} \sin(\omega t) + \vec{v} \cos(\omega t))$$

$$w(\vec{v} | \vec{r}, t) =$$

Free 2D fermions first

Correct answer:

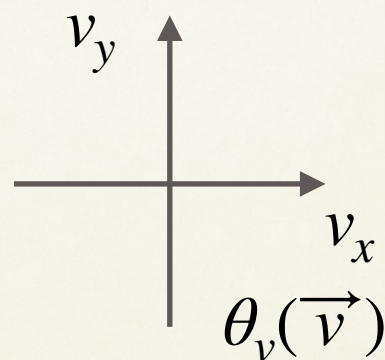
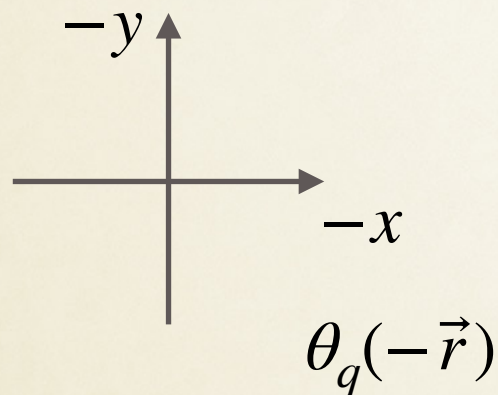


$$\rho(\vec{r}, \vec{v}, t) = \left(\frac{m}{2\pi\hbar} \right)^2 \theta_q(\vec{r} \cos(\omega t) - \vec{v} \sin(\omega t)/\omega) \theta_v(\omega \vec{r} \sin(\omega t) + \vec{v} \cos(\omega t))$$

$$w(\vec{v} | \vec{r}, t) =$$


Free 2D fermions first

Correct answer:



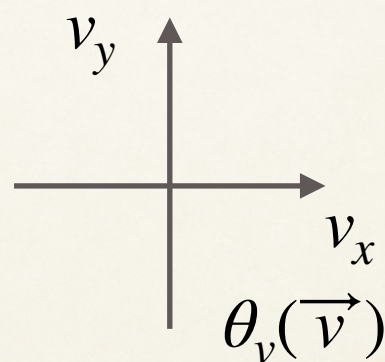
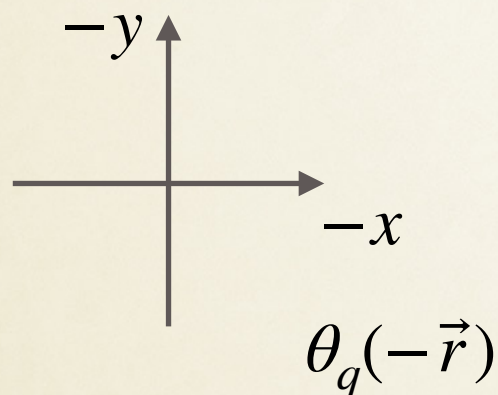
$$\rho(\vec{r}, \vec{v}, t) = \left(\frac{m}{2\pi\hbar} \right)^2 \theta_q(\vec{r} \cos(\omega t) - \vec{v} \sin(\omega t)/\omega) \theta_v(\omega \vec{r} \sin(\omega t) + \vec{v} \cos(\omega t))$$

$$w(\vec{v} | \vec{r}, t) = \text{[Diagram of two overlapping triangles, one shaded]} = \text{[Diagram of a single shaded triangle]} \Rightarrow$$

$$\Rightarrow \frac{\int d^2v (\vec{v} - \bar{\vec{v}})^2 \rho(\vec{r}, \vec{v}, t)}{\int d^2v \rho(\vec{r}, \vec{v}, t)} = \text{const} \Rightarrow \epsilon = \epsilon(n) \frac{2\pi^2}{3\sqrt{3}} \frac{\hbar^2 n}{m} \Rightarrow \mu(n) = \frac{4\pi^2}{3\sqrt{3}} \frac{\hbar^2 n}{m}$$

Free 2D fermions first

Correct answer:



$$\rho(\vec{r}, \vec{v}, t) = \left(\frac{m}{2\pi\hbar} \right)^2 \theta_q(\vec{r} \cos(\omega t) - \vec{v} \sin(\omega t)/\omega) \theta_v(\omega \vec{r} \sin(\omega t) + \vec{v} \cos(\omega t))$$

$$w(\vec{v} | \vec{r}, t) = \text{Venn diagram with two overlapping triangles} = \text{Venn diagram with one triangle} \Rightarrow$$

$$\Rightarrow \frac{\int d^2v (\vec{v} - \bar{\vec{v}})^2 \rho(\vec{r}, \vec{v}, t)}{\int d^2v \rho(\vec{r}, \vec{v}, t)} = \text{const} \Rightarrow \epsilon = \epsilon(n) \frac{2\pi^2}{3\sqrt{3}} \frac{\hbar^2 n}{m} \Rightarrow \mu(n) = \frac{4\pi^2}{3\sqrt{3}} \frac{\hbar^2 n}{m}$$

Cf. standard EoS: $\mu(n) = 2\pi \frac{\hbar^2 n}{m}$

Free 2D fermions first

Reformulated Question:

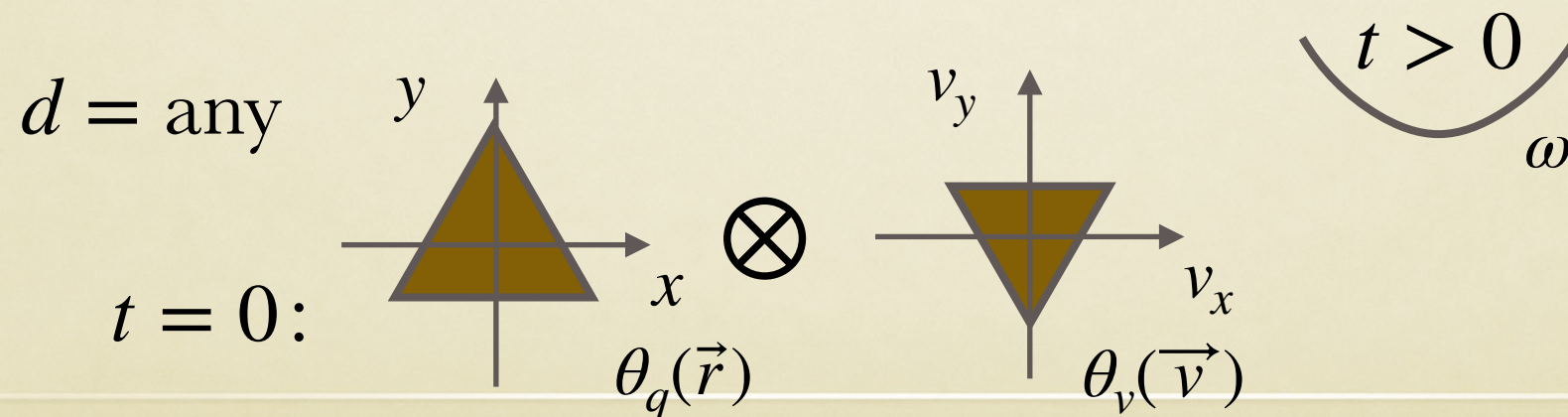
Under what circumstances, velocity distribution of zero-temperature fermions,

$$\rho(\vec{r}, \vec{v}) = \left(\frac{m}{2\pi\hbar} \right)^d \theta(\vec{r}, \vec{v}),$$

remains self-similar at all times.

Answer:

Tensor product of simplexes for $\rho(\vec{r}, \vec{v}, t = 0)$, harmonic trap:



Byproduct: scale-invariant gases

Take

$$\mu(n) = \frac{4\pi^2}{3\sqrt{3}} \frac{\hbar^2 n}{m} .$$

Replace: $\hbar \rightarrow \text{"}\hbar\text{"}$

$$\text{Set: } \frac{4\pi^2}{3\sqrt{3}} \frac{\text{"}\hbar\text{"}^2}{m} \rightarrow g$$

$$\text{Get: } \boxed{\mu(n) = gn}$$

Obtain: exact hydrodynamics of a triangular scale-invariant gas

Reproduces the ENS experiment!

Nonlinear transport
equation:
Chandrasekhar's solution

S. Chandrasekhar, On the decay of plane shock waves,
Ballistic Research Laboratory Report No. 423 (1943).

[https://en.wikisource.org/wiki/
File:Chandrasekhar_-_On_the_decay_Cover.png](https://en.wikisource.org/wiki/File:Chandrasekhar_-_On_the_decay_Cover.png)



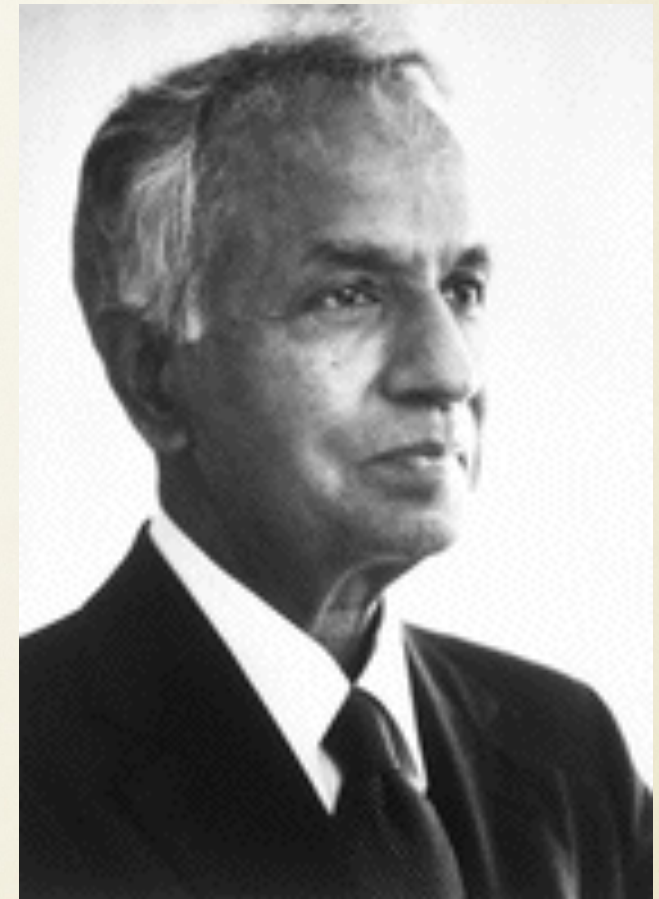
REPORT NO. 423
NOVEMBER 1943

ON THE DECAY OF PLANE SHOCK WAVES

S. Chandrasekhar

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND



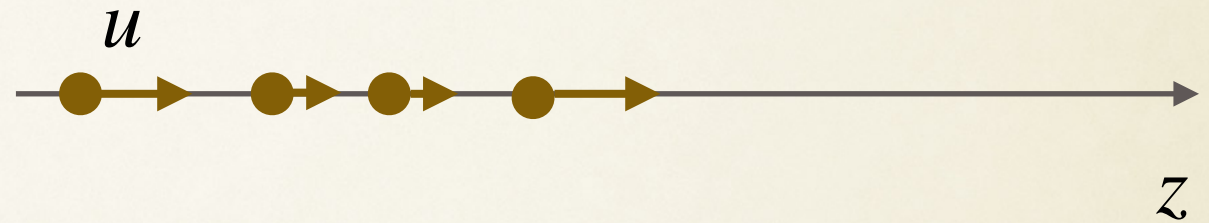
Subrahmanyan
Chandrasekhar

"On the decay of plane Shock Waves", Nov 1943

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial z}u = 0 \quad .$$

General solution, implicit:

$$u = f(z - ut) \quad .$$



Chandrasekhar's solution:

$$u(z, t) = \frac{z}{t} \quad ;$$

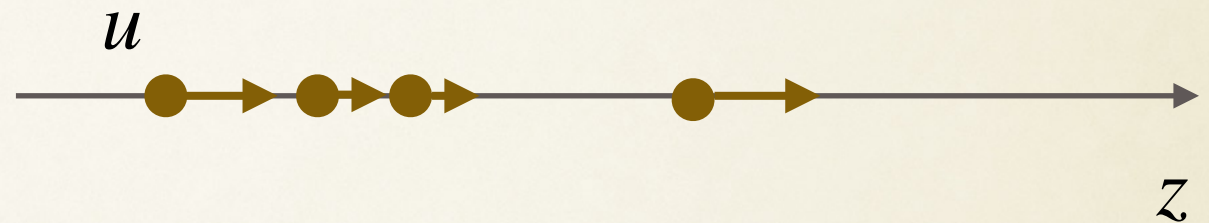
it formally corresponds to

$$f(\xi) = \lim_{\delta t \rightarrow 0} \frac{\xi}{\delta t} \quad .$$

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial z}u = 0 \quad .$$

General solution, implicit:

$$u = f(z - ut) \quad .$$



Chandrasekhar's solution:

$$u(z, t) = \frac{z}{t} \quad ;$$

it formally corresponds to

$$f(\xi) = \lim_{\delta t \rightarrow 0} \frac{\xi}{\delta t} \quad .$$

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial z}u = 0 \quad .$$

General solution, implicit:

$$u = f(z - ut) \quad .$$



Chandrasekhar's solution:

$$u(z, t) = \frac{z}{t} \quad ;$$

it formally corresponds to

$$f(\xi) = \lim_{\delta t \rightarrow 0} \frac{\xi}{\delta t} \quad .$$

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial z}u = 0 \quad .$$

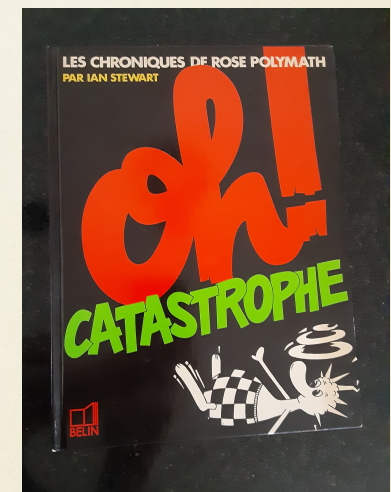
General solution, implicit:

$$u = f(z - ut) \quad .$$



Chandrasekhar's solution:

$$u(z, t) = \frac{z}{t} \quad ;$$



it formally corresponds to

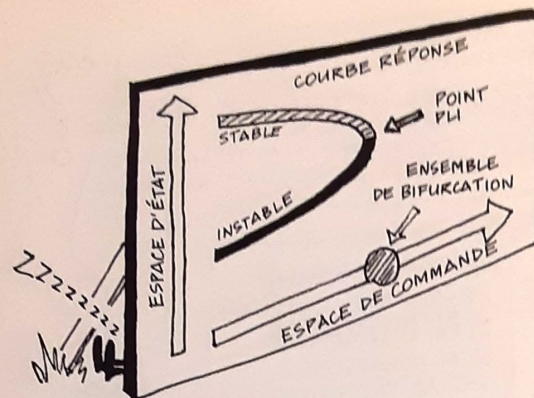
$$f(\xi) = \lim_{\delta t \rightarrow 0} \frac{\xi}{\delta t} \quad .$$

LES CHRONIQUES DE ROSE POLYMATH
PAR IAN STEWART

OK! CATASTROPHE



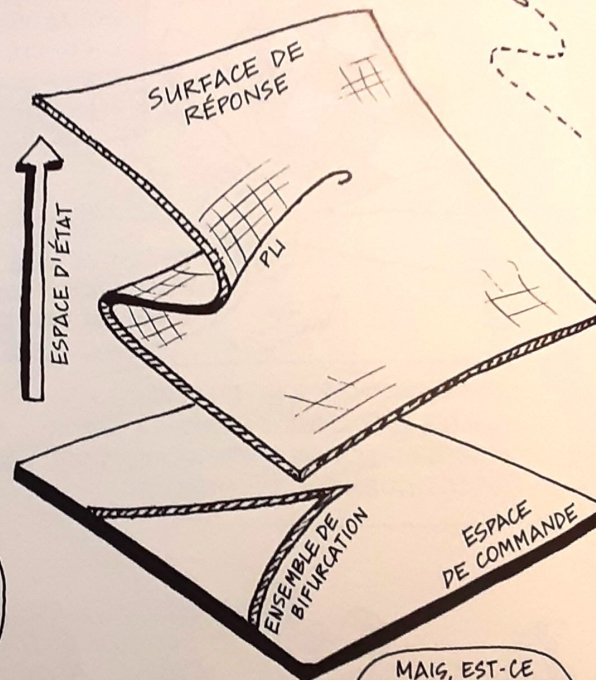

BELIN



POUR LA CATASTROPHE
APPELÉE **PLI**, L'ESPACE
DE COMMANDE EST À UNE
DIMENSION ET L'ENSEMBLE
DE BIFURCATION EST UN POINT.



VOICI ENCORE LA
FRONCE
VUE SOUS UN AUTRE
ANGLE. L'ESPACE
DE COMMANDE
EST À DEUX
DIMENSIONS ET
L'ESPACE DES ÉTATS
EST À UNE DIMENSION.
L'ENSEMBLE DE
BIFURCATION EST UNE
FRONCE ET LE POINT
CATASTROPHIQUE
EST UN POINT FRONCE...
D'OÙ SON NOM.



TU VOIS, GASTON,
LA FRONCE EST UNE CATASTROPHE
**NATURELLE ET
FONDAMENTALE**



GASTON A POSÉ UNE QUESTION
INTÉRESSANTE. EN FAIT, UNE THÉORIE
NE PEUT ÊTRE CONFIRMÉE QUE PAR DES
EXPÉRIENCES. MAIS POUR ÉLABORER
DES EXPÉRIENCES, IL FAUT PARTIR
D'UNE THÉORIE. LA FRONCE EST UNE
THÉORIE **PLAUSIBLE.**

MAIS, EST-CE
QU'ELLE EXPLIQUE
VRAIMENT
QUELQUE CHOSE?



ENSUITE
LA **QUE**
3 DIMEN
ET UNE
DESSINER
C'ES
AUSSI JE

LES CHRONIQUES DE ROSE POLYMATH
PAR IAN STEWART

CATASTROPHE

BELIN

What is Mathematics?

SECOND EDITION

An
Elementary
Approach to
Ideas and
Methods

Richard Courant and Herbert Robbins
Revised by Ian Stewart

"A lucid representation of the fundamental concepts and methods of the whole field of mathematics."
—Albert Einstein



Damski map:
from the nonlinear transport
equation to GPE
hydrodynamics

B. Damski, Formation of shock waves in a Bose-Einstein condensate, Phys.Rev. A 69, 043610 (2004).

1D GPE hydrodynamics:

$$\frac{\partial}{\partial t} n_{1D} + \frac{\partial}{\partial z} (n_{1D} v_{1D}) = 0$$

$$\frac{\partial}{\partial t} v_{1D} + \left(v_{1D} \frac{\partial}{\partial z} \right) v_{1D} = -\frac{1}{m} \frac{\partial}{\partial z} (g_{1D} n_{1D}) .$$

Damski map:

$$n_{1D} = \frac{1}{9} \frac{m}{g_{1D}} u^2$$

$$v_{1D} = \frac{2}{3} u$$

, plus Galilean boosts.

Nonlinear Transport Equation:

$$\frac{\partial}{\partial t} u + u \frac{\partial}{\partial z} u = 0 .$$

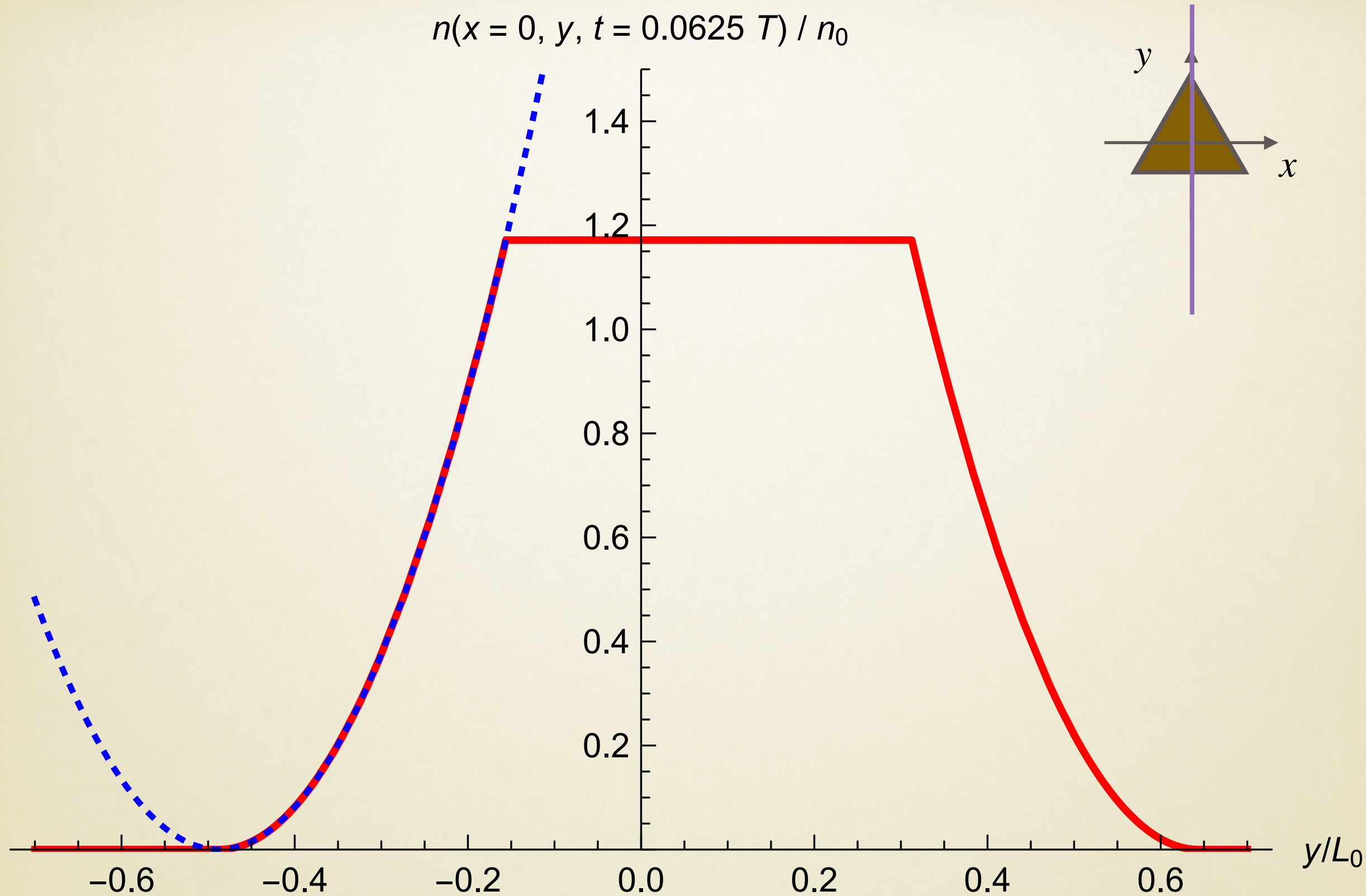
The $SO(2, 1)$ map from the
free to the trapped GPE
hydrodynamics

L. P. Pitaevskii and A. Rosch, Breathing modes and hidden symmetry of trapped atoms in two dimensions, Phys. Rev. A 55(2), R853 (1997).

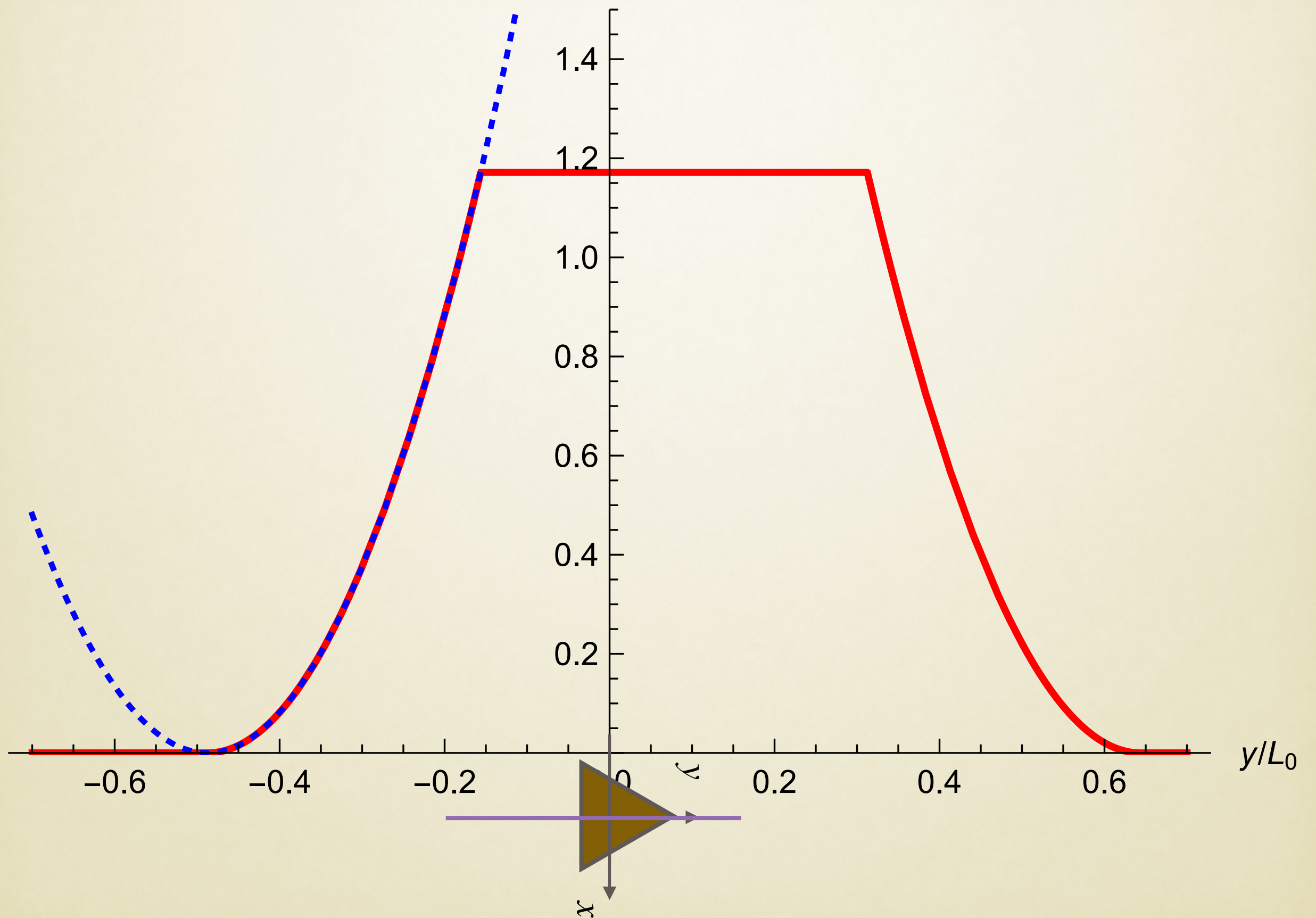
...

Connecting the ends:
Damski-Chandrasekhar
shockwave vs. a triangular
breather

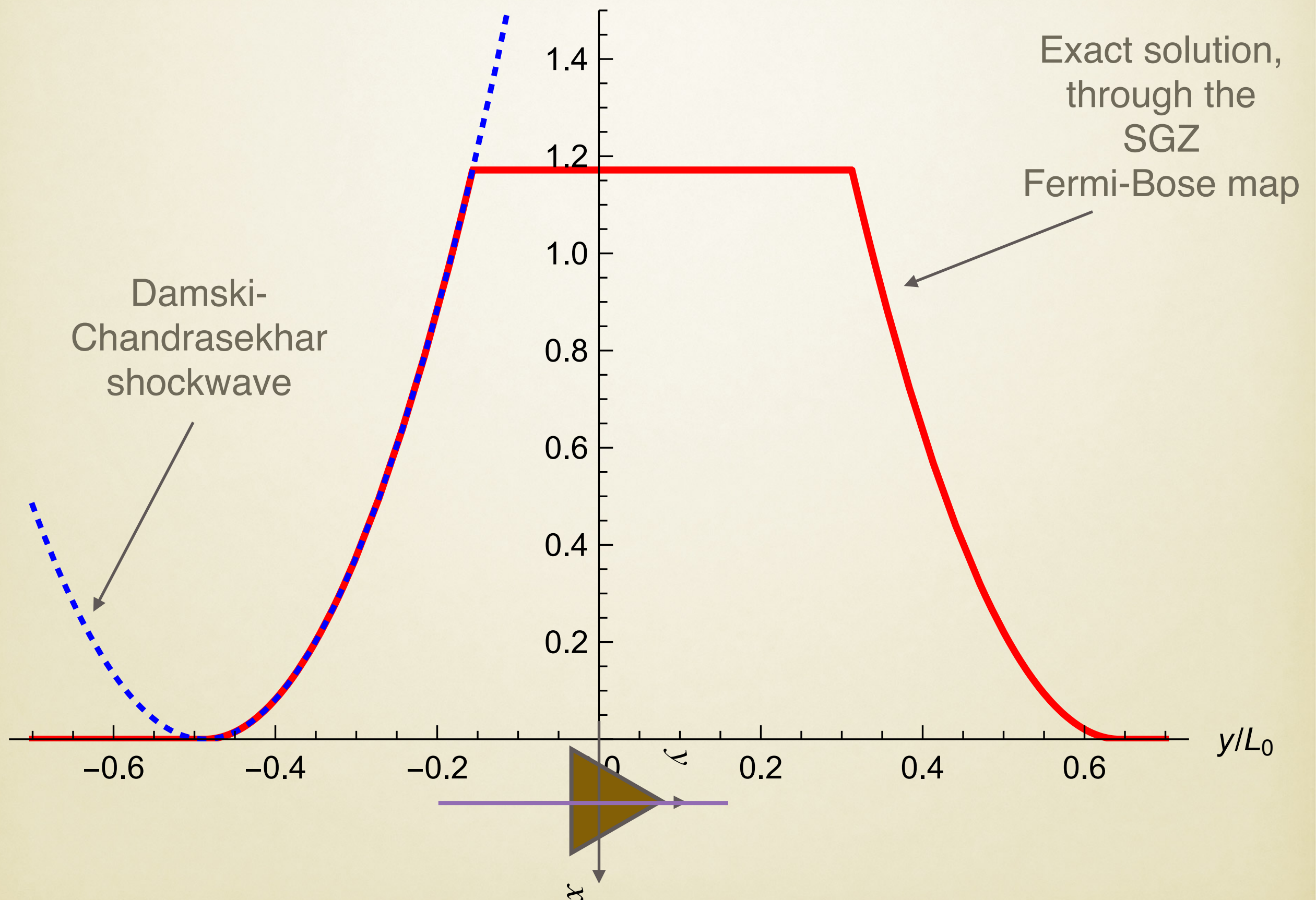
Triangular Gross-Pitaevskii breathers and Damski-Chandrasekhar shock waves, M. Olshanii, D. Deshommes, J. Torrents, M. Gonchenko, V. Dunjko, G. E. Astrakharchik, SciPost Phys. 10, 114 (2021).



$$n(x = 0, y, t = 0.0625 T) / n_0$$



$$n(x = 0, y, t = 0.0625 T) / n_0$$



Why Damski-Chandrasekhar
shockwave and nothing else
emanates from a sharp
density step in GPE?

We don't know yet

One thing we do know is that
there is *no*
beyond-hydrodynamics energy
scales.

If one can prove that
Damaski-Chandrasekhar is
"unique" in some sense, this
will answer the question.

Can features of the order of the healing length lead to infinities from the hydrodynamical perspective?

Hydrodynamical energy scale:

$$E_{\text{HD}} \sim \mu N$$

Acceleration energy:

$$E_{\text{acceleration}} \sim \mu n \xi R \sim \frac{\xi}{R} E_{\text{HD}} \ll E_{\text{HD}} .$$

Heisenberg uncertainty energy:

$$E_{\text{Heisenberg}} \sim \frac{\hbar^2}{m \xi^2} n R \xi \sim \mu n \xi R \sim \frac{\xi}{R} E_{\text{HD}} \ll E_{\text{HD}} .$$

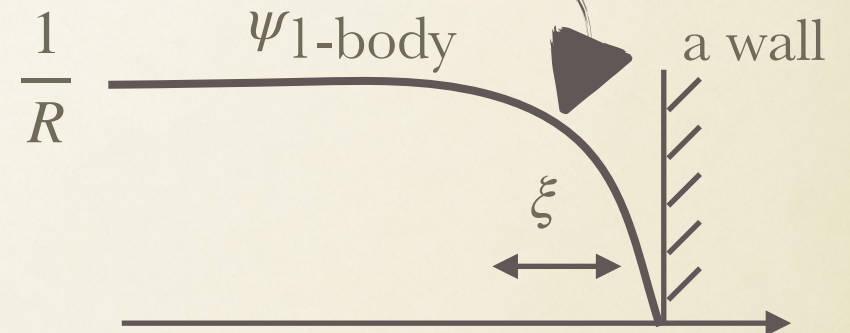
Cusp energy:

$$w(\vec{p}) \sim \frac{\hbar^2}{\xi^2} \frac{1}{p^4}$$

$$p_{\text{cut-off across}} \sim \frac{\hbar}{\xi} ; \quad p_{\text{width along}} \sim \frac{\hbar}{R}$$

$$E_{\text{cusp}} \sim \int \int d^2 \vec{p} \frac{p^2}{m} w(\vec{p}) N \sim p_{\text{width along}} \frac{\hbar^2}{m \xi^2} \frac{1}{p} \Big|_{p \sim p_{\text{cut-off across}}} N \sim \frac{\xi}{R} E_{\text{HD}} \ll E_{\text{HD}} .$$

2D + wall



Can features of the order of the healing length lead to infinities from the hydrodynamical perspective?

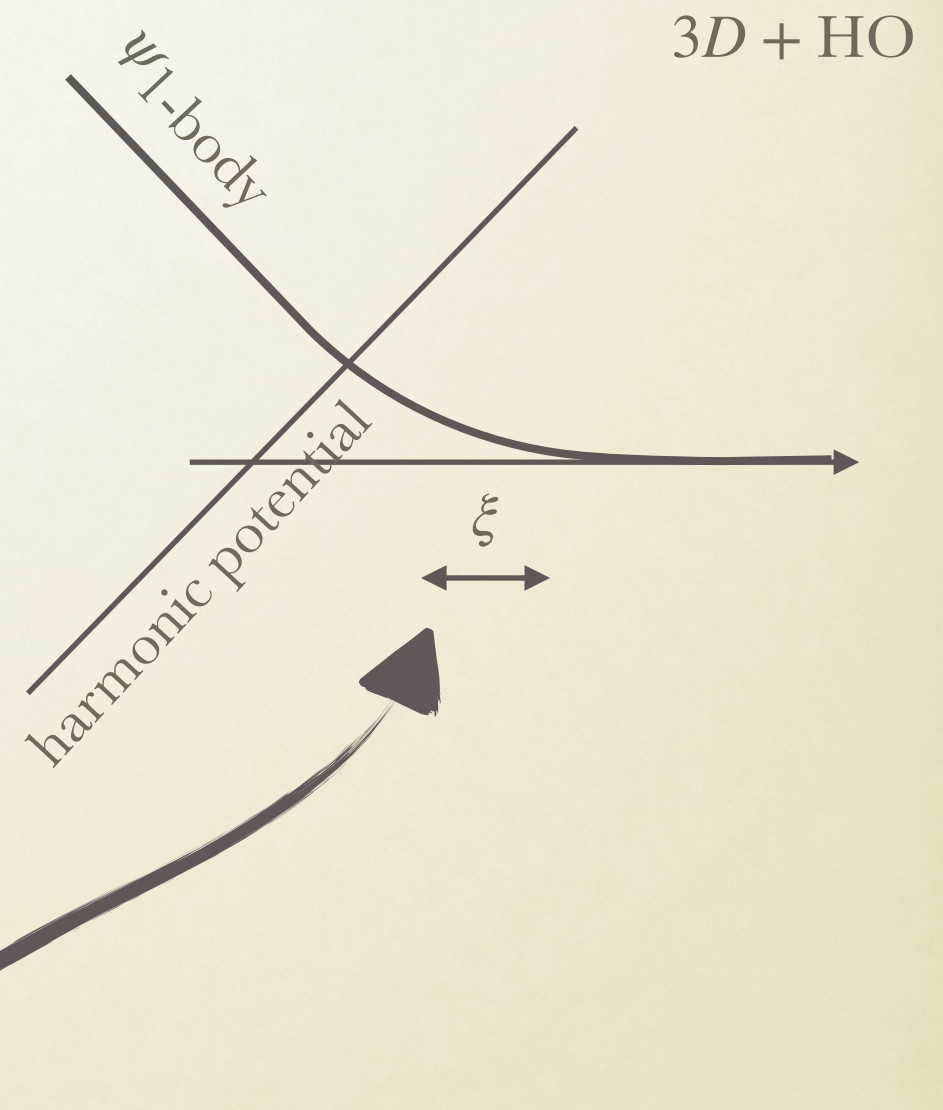
Cf. F. Dalfovo, L. Pitaevskii, S. Stringari,
"Order Parameter at the Boundary of a Trapped Bose Gas",
Phys. Rev. **A 54**, 4213 (1996)

Hydrodynamical energy scale:

$$E_{\text{HD}} \sim \mu N .$$

Cusp energy:

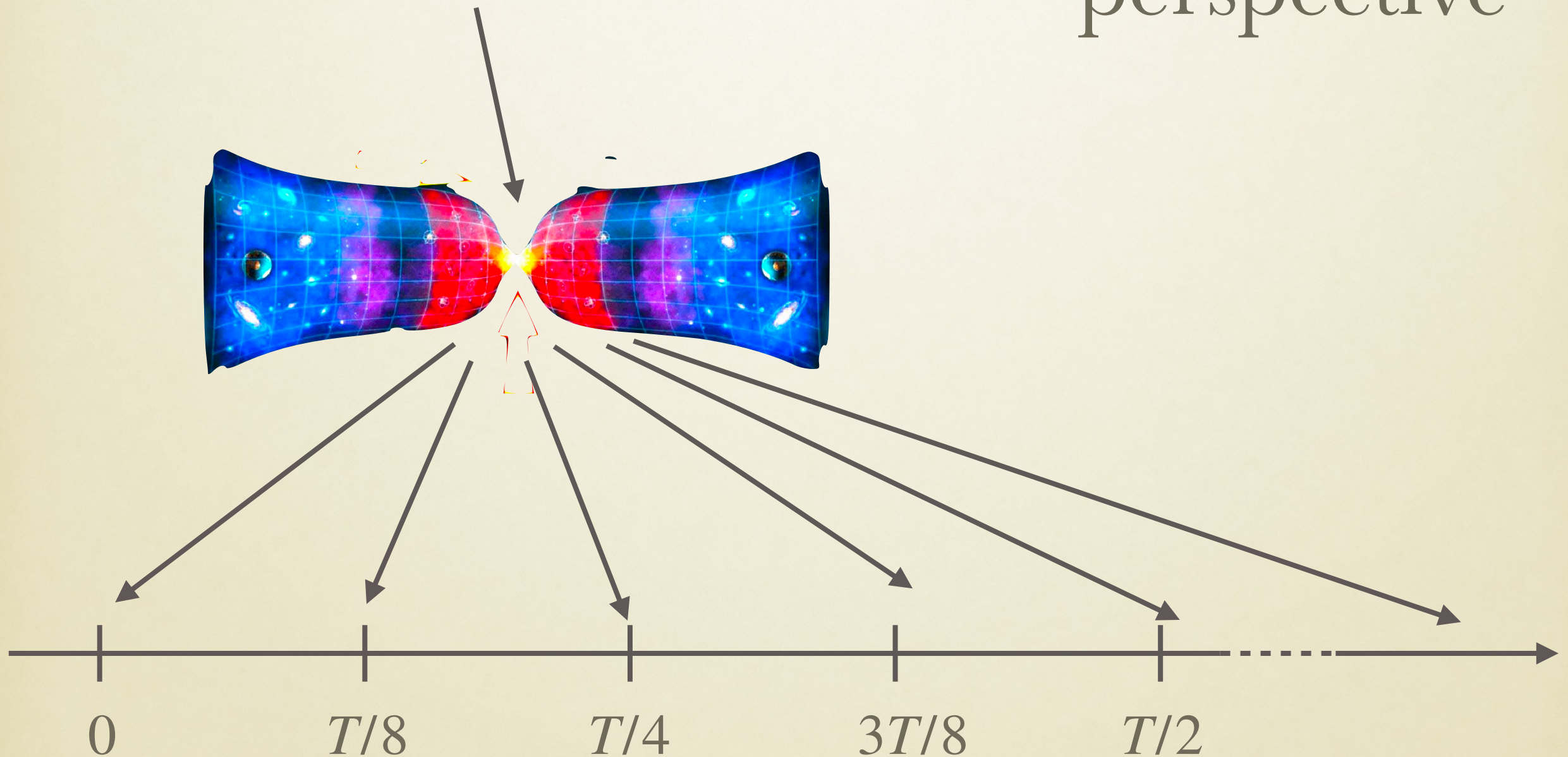
$$E_{\text{cusp}} \sim \left(\frac{\xi}{R} \right)^2 \ln \left(\frac{R}{\xi} \right) E_{\text{HD}} \ll E_{\text{HD}} .$$



Why GPE triangles are special: hydrodynamics perspective

a "Big Crunch/Big Bang" catastrophe at
the hydrodynamics level:

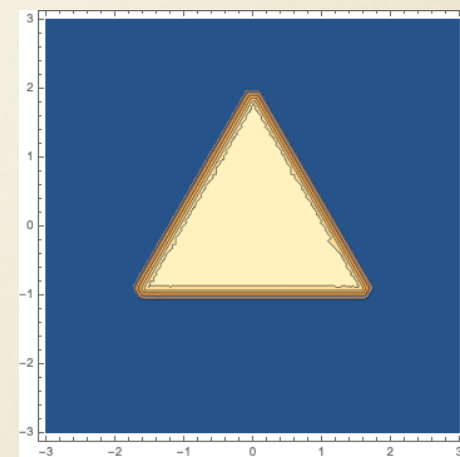
a discontinuity in n_t and $\vec{v}_t$ that
is controlled by a "GUT" theory
above (GPE, in our case)



Why GPE triangles are special: hydrodynamics perspective

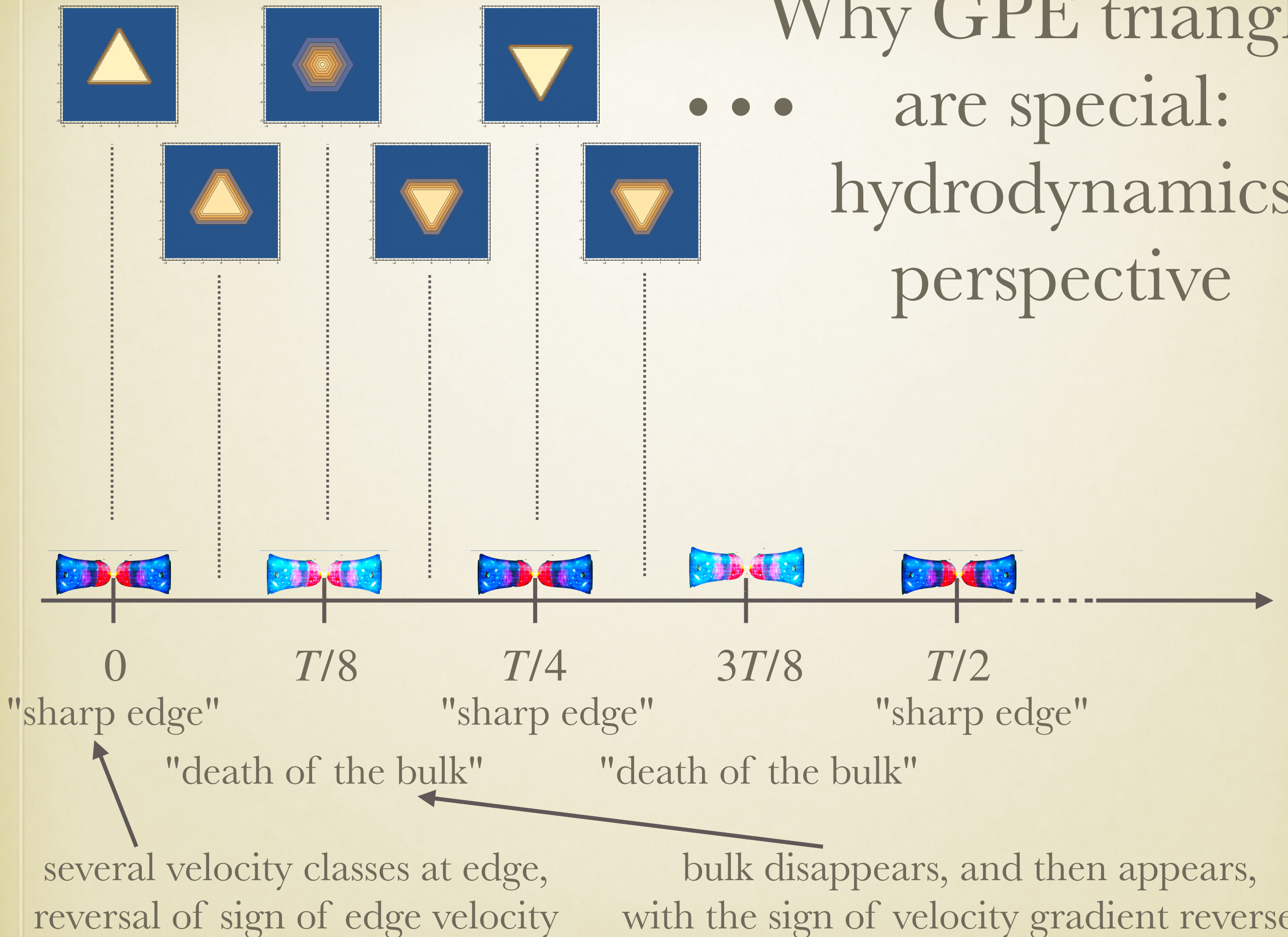
a "Big Crunch/Big Bang" catastrophe at
the hydrodynamics level:

a discontinuity in n_t and $\vec{v}_t$ that
is controlled by a "GUT" theory
above (GPE, in our case)

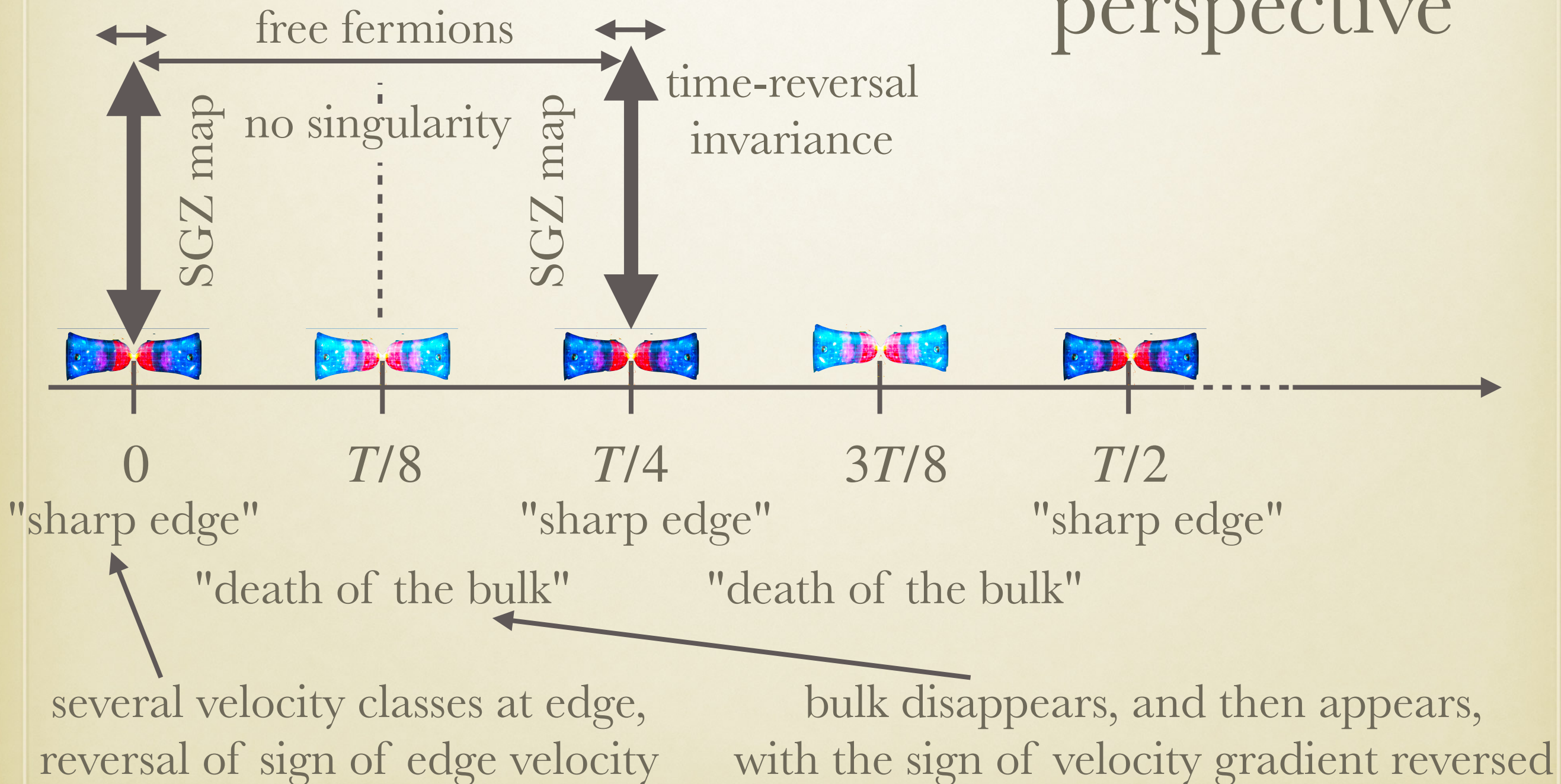
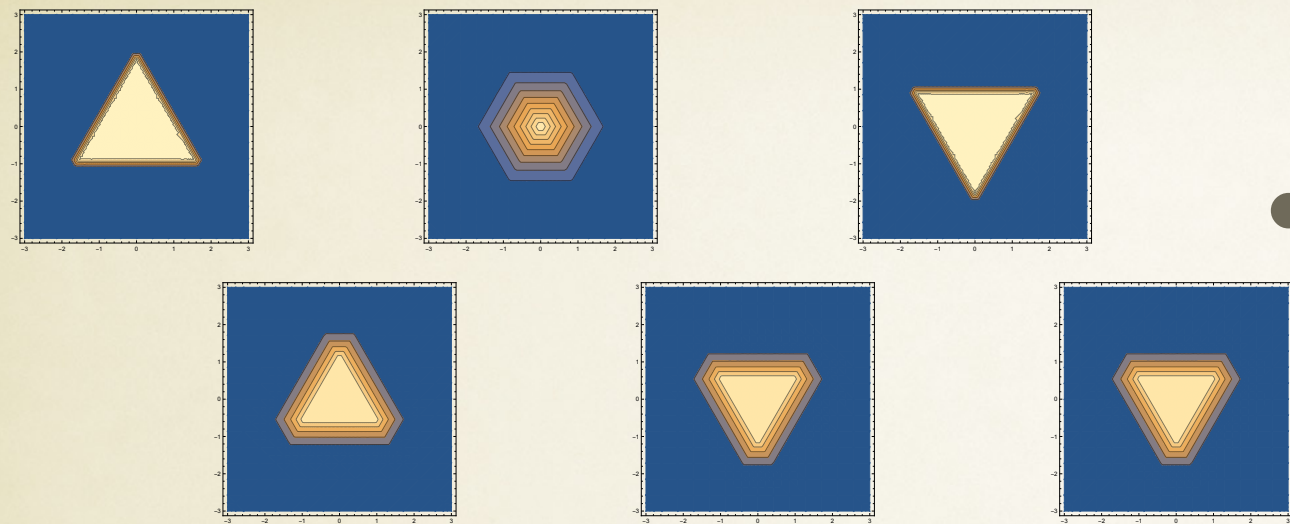


Why GPE triangles

are special:
hydrodynamics
perspective



Why GPE triangles are special: hydrodynamics perspective

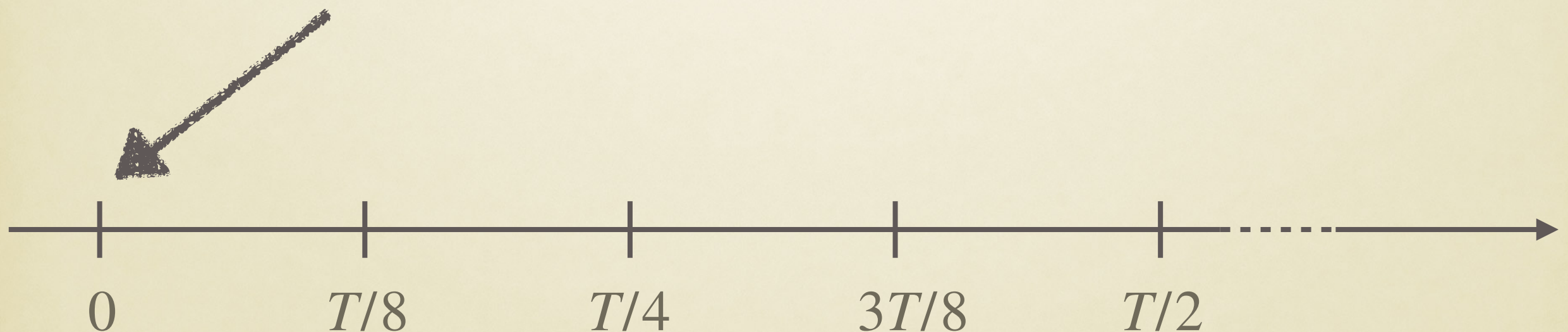
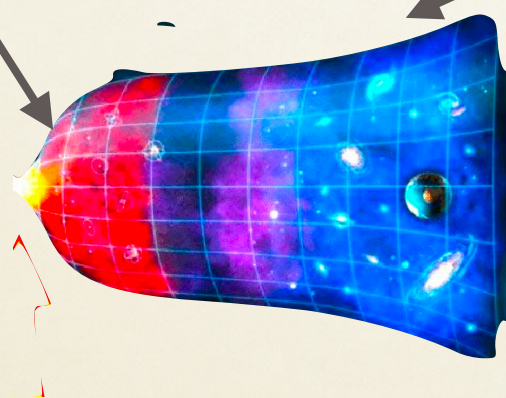


"First three minutes"

Triangular Gross-Pitaevskii breathers and Damski-Chandrasekhar shock waves, M. Olshanii, D. Deshommes, J. Torrents, M. Gonchenko, V. Dunjko, G. E. Astrakharchik, SciPost Phys. 10, 114 (2021).

"Universe as we know it"

Z.-Y. Shi, C. Gao and H. Zhai, Idealized hydrodynamics (2020), arXiv/2011.01415.



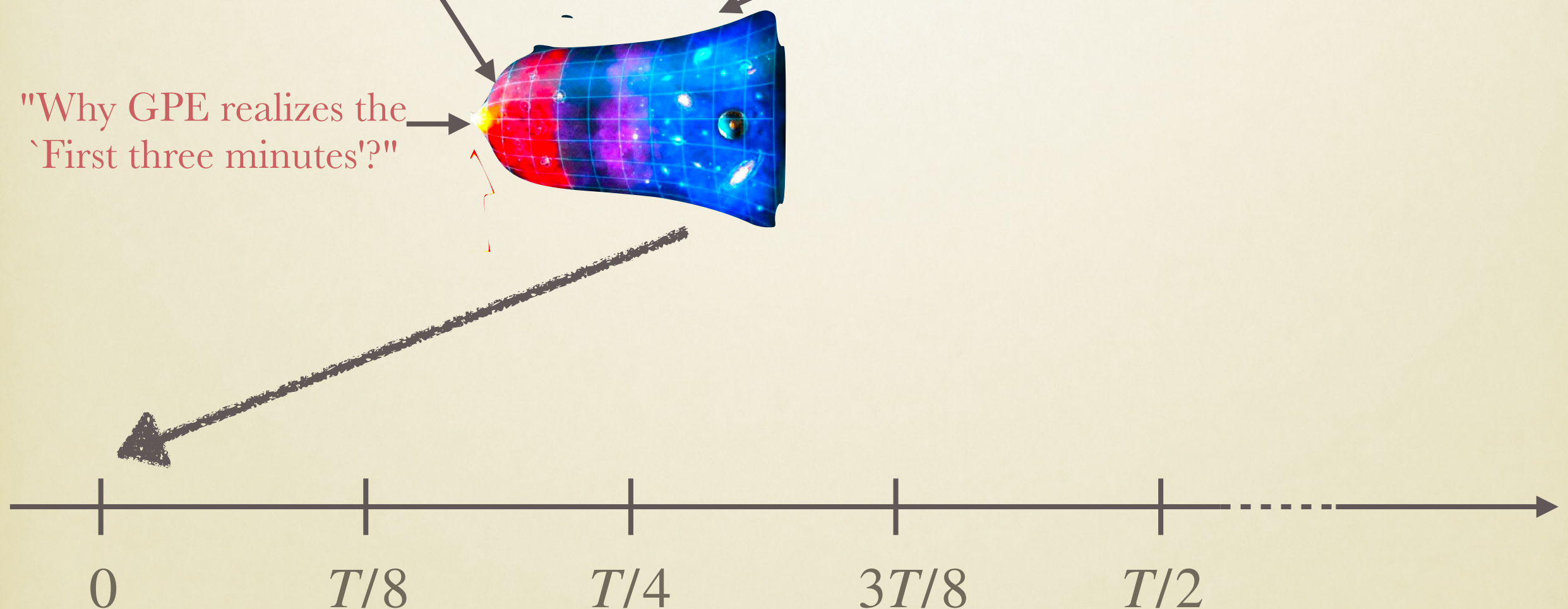
"First three minutes"

Triangular Gross-Pitaevskii breathers and Damski-Chandrasekhar shock waves, M. Olshanii, D. Deshommes, J. Torrents, M. Gonchenko, V. Dunjko, G. E. Astrakharchik, SciPost Phys. 10, 114 (2021).

"Universe as we know it"

Z.-Y. Shi, C. Gao and H. Zhai, Idealized hydrodynamics (2020), arXiv/2011.01415.

"Why GPE realizes the
'First three minutes'?"



Triangular Gross-Pitaevskii breathers and Damski-Chandrasekhar shock waves, M. Olshanii, D. Deshommes, J. Torrents, M. Gonchenko, V. Dunjko, G. E. Astrakharchik, SciPost Phys. 10, 114 (2021).

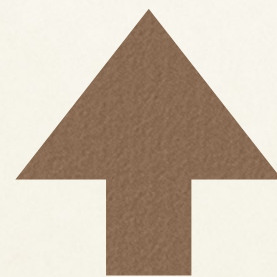
Summary

Trapped triangular GPE breathers
[Saint-Jalm, Dalibard, et al (2019)]

Expalnation for triang. breathers:
interacting bosons $\leftrightarrow$ free fermions
map; simplexes only, scale-
invariant interactions only, both
trapped [Shi-Gao-Zhai (2020)]



$SO(2, 1)$ map to untrapped particles [Pitaevskii-Rosch (1997)]



Map to a Nonlinear Transport Equation [Damski (2004)]



Chandrasekhar's solution of the Nonlinear Transport
Equation [Chandrasekhar (1943)]

Outlook

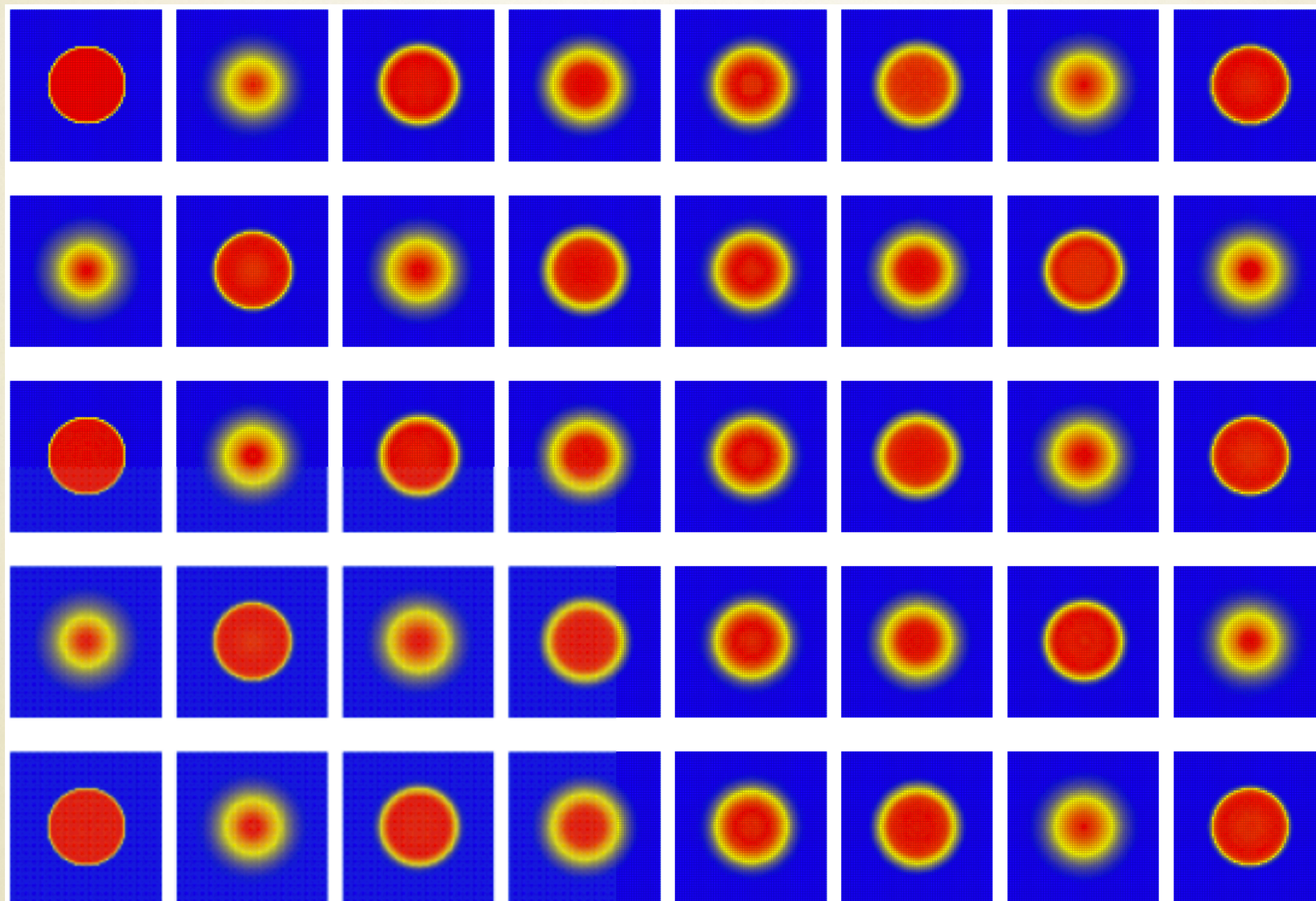
Open questions in triangular breathers:

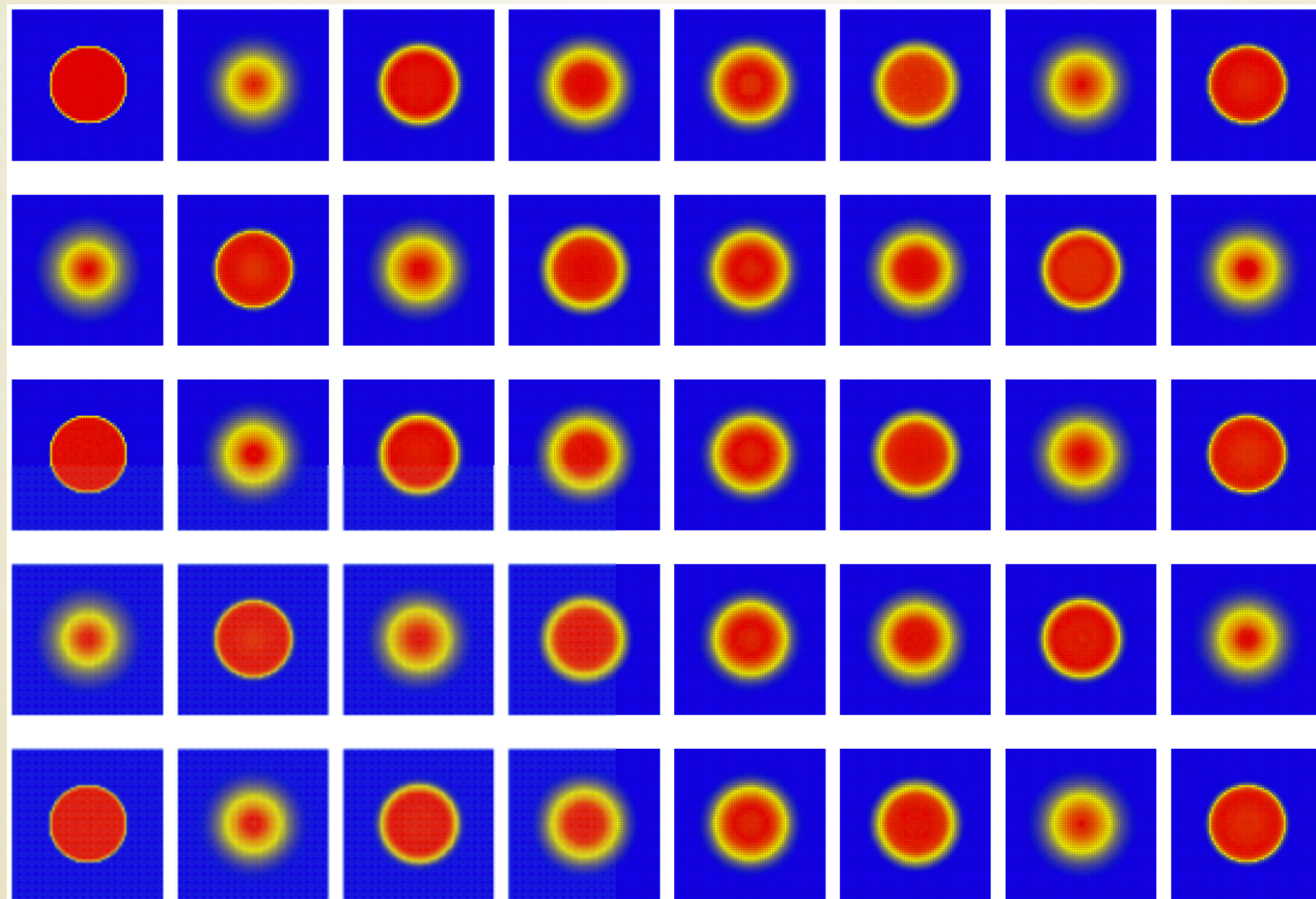
(*) Why GPE decides to realize Damski-Chandrasekhar's catastrophe at $t = 0$ and nothing else?

(*) Why equilateral triangles make it through the "death of the bulk" catastrophe at $t = T/8$, but other don't?

Both are supported by the Bose-Fermi map. Other polygons aren't.

Next summit:
disk breathers





The story is much more convoluted!

Support by:



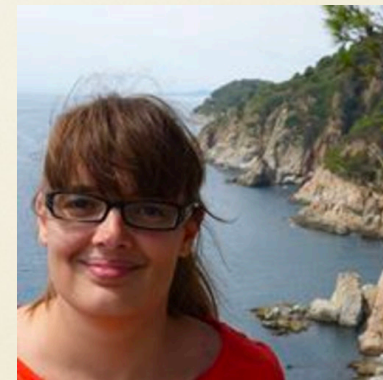
"Triangular Gross-Pitaevskii breathers and Damski-Chandrasekhar shock waves", M. Olshanii, D. Deshommes, J. Torrents, M. Gonchenko, V. Dunjko, G. E. Astrakharchik, [arXiv:2102.12184](#)

Dumesle Deshommes (UMass Boston)

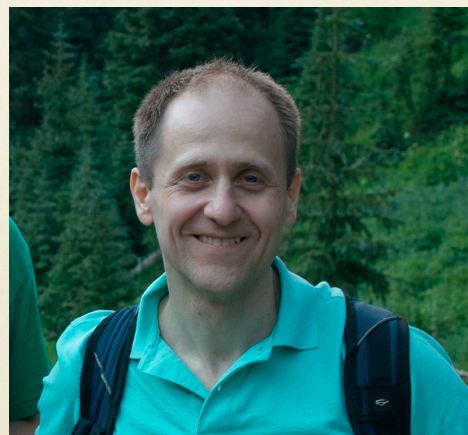


Jordi Torrents (U Barcelona)

Marina Gonchenko (U Politècnica de Catalunya)

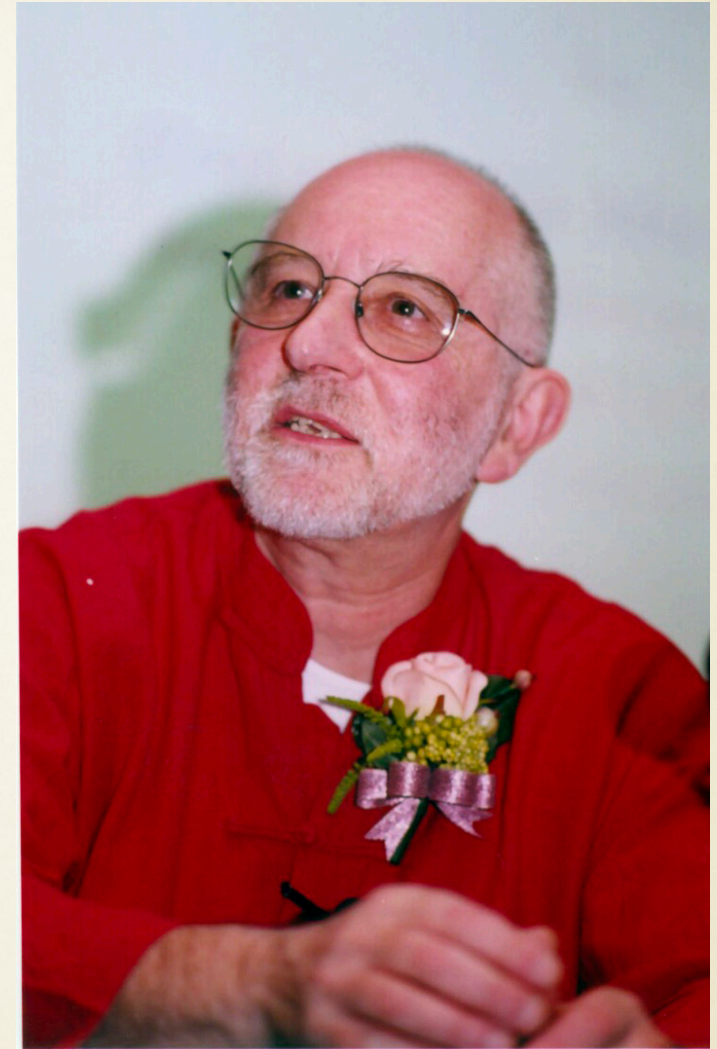


Vanja Dunjko (UMass Boston)



Gregory Astrakharchik (U Politècnica de Catalunya)





“Nothing is ever discovered for the first time.”

-- Berry's Law

Thank you!