Anyons from Three-Body Hard-Core Interactions in One Dimension


Nathan Harshman
Department of Physics, American University
Washington, DC, USA
2018 July 4
Goals for this talk

• Introduce traid anyons
  – possible in one-dimension when there are hard-core three-body interactions, arbitrary two-body interactions (not hard-core)
  – traid anyons are particles with generalized exchange statistics similar to more familiar braid anyons, but different

• Advertise configuration space methods for few-body physics
  – geometry, symmetry, and topology

• Motivate study of solvable models
  – separability, integrability, entanglement, solvability, control
Physical Motivation

• Effective one-dimensional particle models in cold atoms systems
  – Experiments with traps and lattices
  – Hard-core, two-body interactions

• Effective three-body interactions in cold atom systems
  – Proposals for traps (including rotating traps) and lattices
  – Hard-core, three-body interactions?

• New few-body physics, new many-body physics
  – New mathematical physics
Previous Work


- Many-body physics and thermodynamics

- Few-body physics

What are anyons?
Normal Exchange Statistics:
particle permutations of identical particles given by symmetric group

\[ p \in S_N \]

- **Bosons**
  \[
  \hat{U}(p)\psi(x_1, x_2, \ldots, x_N) = \psi(x_{p_1}, x_{p_2}, \ldots, x_{p_N}) = \psi(x_1, x_2, \ldots, x_N)
  \]

- **Fermions**
  \[
  \hat{U}(p)\psi(x_1, x_2, \ldots, x_N) = \begin{cases} 
  \psi(x_1, x_2, \ldots, x_N) & p \in \text{even} \\
  -\psi(x_1, x_2, \ldots, x_N) & p \in \text{odd}
  \end{cases}
  \]

- **Parastatistics** (useful for partially distinguishable identical particles)
  \[
  \hat{U}(p)\psi_i(x_1, x_2, \ldots, x_N) = \sum_j D_{ij}(p)\psi_j(x_1, x_2, \ldots, x_N)
  \]

Anyons do *NOT* have

\[ \text{abelian} \]
\[ \text{non- abelian} \]
Generalized Exchange Statistics

• Group describing particle exchanges is \(^{\text{NOT}}\) the symmetric group.
  – Most famous example: braid group and fractional exchange statistics
  – One-dimensional, hard-core three-body anyons: \(^{\text{traid}}\) group

• Transformation of wave function depends not just on which particles were exchanged, but on \(^{\text{HOW}}\) they were exchanged.
  – Transformation depends on path in configuration space, so topology of configuration space matters.
  – There may even be a non-trivial transformation when particles undergo an adiabatic transformation back to initial configuration, aka Berry phase.
Crash Course on Braid Anyons, Part I

• Hard-core (or singular) two-body interactions in two dimensions
• Consider an exchange in...

![Diagram of space, relative configuration space, and path space]
Crash Course on Braid Anyons, Part II

• Not all paths connecting same configurations are equivalent

\[ b_1 \neq b_1^{-1} \]

\[ (b_1)^2 \neq 1 \neq (b_1^{-1})^2 \]
Crash Course on Braid Anyons, Part III

- Generalize to $N$ particles, $N$ strands

\[ b_1 b_2 b_1 = b_2 b_1 b_2 \]

Braid relation

aka third Reidemeister move or Yang-Baxter relation

- Braid group

\[ B_N = \langle b_1, b_2, \ldots, b_{N-1} \mid b_ib_{i+1}b_i = b_{i+1}b_i b_i; b_ib_j = b_jb_i \text{ when } j \geq i + 2 \rangle \]

generators

braid relation
distant braids commute
Crash Course on Braid Anyons, Part IV

• Fractional exchange statistics
  – Abelian representation of braid group characterized by parameter:

\[
\hat{U}(b_i)\psi(x_1, x_2, \ldots) = e^{i\pi\nu\hat{\mathcal{P}}_{b_i}}\psi(x_1, x_2, \ldots) \quad \nu \begin{cases} 
  = 0 & \text{bosons} \\
  = 1 & \text{fermions} \\
  \in (0, 1) & \text{braid anyons}
\end{cases}
\]

  – Leads to either multivalued wave function or single-valued wave function and gauge potential

• Fractional exclusion statistics
  – Inverse parameter quantifies identical particles per state

• Non-abelian representations, too
Generalized Exchange Statistics: occur when configuration space *not simply connected*

• Three reasons this can occur:
  1. Underlying space has non-trivial topology (e.g. wells, ring, torus)
  2. Hard-core interactions create defects in configuration space
  3. Indistinguishable particles induce non-trivial topology on configuration space
Crash Course on Braid Anyons, Part V

• Configuration space for
  – \( N \) particles
  – in \( d=2 \)-dimensional Euclidean space
  – with \( k=2 \)-body hard-core interactions
• NOT simply connected
  – even for distinguishable particles

\[
\mathcal{X}_{N,2,2} \equiv \mathbb{R}^{2N} - \mathcal{V}_{N,2,2}
\]
Generalized Exchange Statistics: occur when configuration space *not simply connected*

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\[ \chi_{N,d,k} \equiv \mathbb{R}^{Nd} - \mathcal{V}_{N,d,k} \]
# Topology of Trapped, Hard Core Particles

Configuration space is

- disconnected: defects like line in plane, planes in space
- connected, but not simply connected: defects like point in plane, lines in space
- everything else, simply connected: defects like point in space

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<th>4-body</th>
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<td>$d = 3$</td>
<td>3</td>
<td>6</td>
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Defect co-dimension does *NOT* depend on $N$.
Configuration space for three 1-D hard-core systems

Two-body hard-core

Six disconnected sectors: equivalent to six points

Three-body hard-core

Connected, but not simply connected: equivalent to a circle
Configuration space for three 1-D hard-core systems

Two-body hard-core

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Three-body hard-core

Connected, but not simply connected: equivalent to a circle
relative configuration space

\[ z_1 = \frac{1}{\sqrt{2}}(x_1 - x_2) \]
\[ z_2 = \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3) \]
Relative configuration space for four 1-D hard-core systems

Two-body hard-core

Three-body hard-core

Four-body hard-core

Twenty-four disconnected sectors: equivalent to 24 points

Connected, but not simply: equivalent to sphere with eight holes

Simply connected: equivalent to sphere
Relative configuration space for four 1-D hard-core systems

Two-body hard-core

Three-body hard-core

Four-body hard-core

Twenty-four disconnected sectors: equivalent to 24 points

Connected, but not simply: equivalent to sphere with eight holes

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Generalized Exchange Statistics:

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• Three reasons this can occur:
  1. Underlying space has non-trivial topology (e.g. wells, ring, torus)
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\[
\chi_{N,d,k}^{N,d} \equiv \frac{\mathbb{R}^{Nd} - \mathcal{V}_{N,d,k}}{S_N}
\]
Generalized Exchange Statistics: occur when configuration space *not simply connected*

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- Fundamental group of configuration space:
  - Defined as maps among homotopy-equivalent classes of paths
  - Describes particle exchange symmetry
  - Irreducible representations with GES are anyons
Fundamental groups

- **Braid Group**
  \[ B_N = \pi_1 \left( \frac{\mathbb{R}^{2N} - \mathcal{V}_{N,2,2}}{S_N} \right) \]
  - Pure braid group
  \[ PB_N = \pi_1 \left( \mathbb{R}^{2N} - \mathcal{V}_{N,2,2} \right) \]

- **Traid Group**
  \[ T_N = \pi^*_1 \left( \frac{\mathbb{R}^{N} - \mathcal{V}_{N,1,3}}{S_N} \right) \]
  - Pure traid group
  \[ PT_N = \pi^*_1 \left( \mathbb{R}^{N} - \mathcal{V}_{N,1,3} \right) \]

* orbifold fundamental group

- Two dimensions and hard-core two-body interactions
- One dimension and hard-core three-body interactions

- Indistinguishable particles
- Distinguishable particles
Breaking the Symmetric Group

- Symmetric group $S_N$
  - Generators $s_1, s_2, \ldots, s_{N-1}$
  - Relations $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$, $s_is_j = s_js_i$ when $j \geq i + 2$, $s_i^2 = 1$

Braid group

$B_N = \langle b_1, b_2, \ldots, b_{N-1} \mid b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1}; b_ib_j = b_jb_i$ when $j \geq i + 2 \rangle$

Traid group

$T_N = \langle t_1, t_2, \ldots, t_{N-1} \mid t_i^2 = 1; t_it_j = t_jt_i$ when $j \geq i + 2 \rangle$
symmetric group
\[ s_1 = s_1^{-1} \]
\[ s_1 s_2 s_1 = s_2 s_1 s_2 \]

braid group
\[ b_1 \neq b_1^{-1} \]
\[ b_1 b_2 b_1 = b_2 b_1 b_2 \]

traid group
\[ t_1 = t_1^{-1} \]
\[ t_1 t_2 t_1 \neq t_2 t_1 t_2 \]
Traid Group is a Hyperbolic Coxeter Group

• Finite Coxeter groups
  – Tile the sphere under reflections
  – In 3D: tetrahedral, octahedral, icosahedral

• Affine Coxeter groups
  – Tile the flat plane under reflections
  – In 2D: three possibilities

• Hyperbolic Coxeter groups
  – Tile the hyperbolic plane under reflections
  – Classified...I think ????

https://commons.wikimedia.org/wiki/File:H2checkers_2ii.png
One-dimensional few-body models

- Calogero-Moser model
- Calogero-Sutherland model
- elliptic CSM model
- hyperbolic CSM model
- Ruijsenaars-Schneider model
- KdV equation
- Gross-Pitaevski equation
- Jordan-Wigner transformation
- Luttinger liquid
- Heisenberg spin-chain models
- Bose-Hubbard models
- Fermi-Hubbard models
- Lieb-Liniger model
- Kronig-Penney model
- lattice models
- Bethe ansatz
- contact interactions
- unitary limit
- Bose-Hubbard models
- Fermi-Hubbard models
- harmonic models
- one-dimensional braids
- anyons
- hybrid CM-contact model
- one-dimensional braids
- anyons
- quantum wires
- Toda model
- quantum Newton’s Cradle
- Tonks-Girardeau gas
- super-Tonks-Girardeau gas
- quantum anomalies
- mixed mass Coxeter models
- Boson-Fermion mapping

References:
- Andersen, Harshman, Zinner, PRA (2017)
- Harshman, et al., PRX (2017)
## Compare the Anyons

### Braid Anyons
- Hard-core **two-body** interactions in **two dimensions** make co-dimension two defects in configuration space
- GES given by fundamental group of configuration space
- Braid group generalization of symmetric group that breaks **self-inverse relation**
- Abelian and non-abelian representations
- Statistics can be absorbed into **two-body and three-body** gauge potentials
- Breaks time reversal symmetry and breaks space reflection symmetry

### Traid Anyons
- Hard-core **three-body** interactions in **one dimension** make co-dimension two defects in configuration space
- GES given by **orbifold** fundamental group of configuration space
- Traid group generalization of symmetric group that breaks **Yang-Baxter relation**
- Abelian and non-abelian representations
- Statistics can be absorbed into **three-, four-, and five-body** gauge potentials
- Breaks time reversal, but **does not break space reflection symmetry**
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Thanks: M.-A. Garciá-March, A.C. Knapp, M. Olshani, A. Volosniev, and N.T. Zinner