Abandoned symmetry

Vladimir A. Yurovsky
Tel Aviv University

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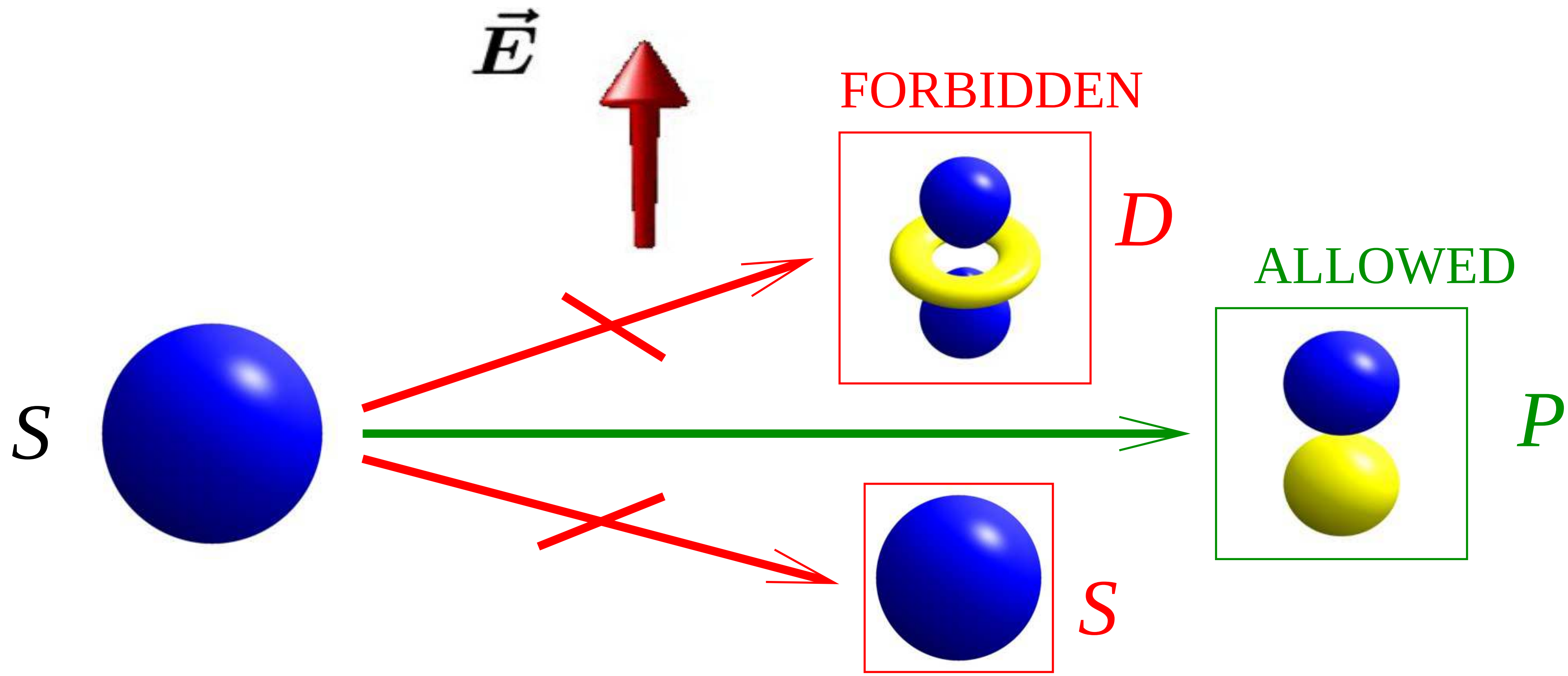
ABANDONED SYMMETRY

SELECTION RULES

Symmetric system + non-symmetric perturbation



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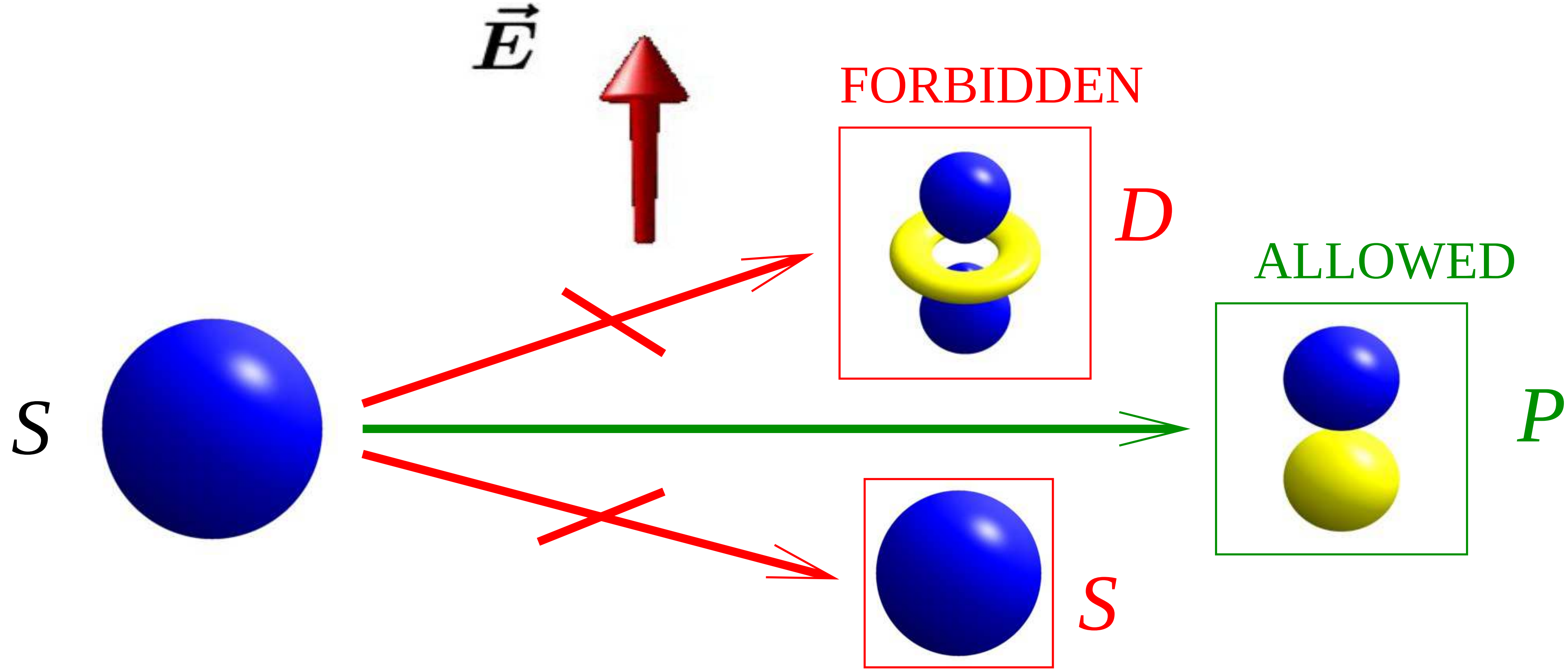

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— **the Pauli exclusion principle.**

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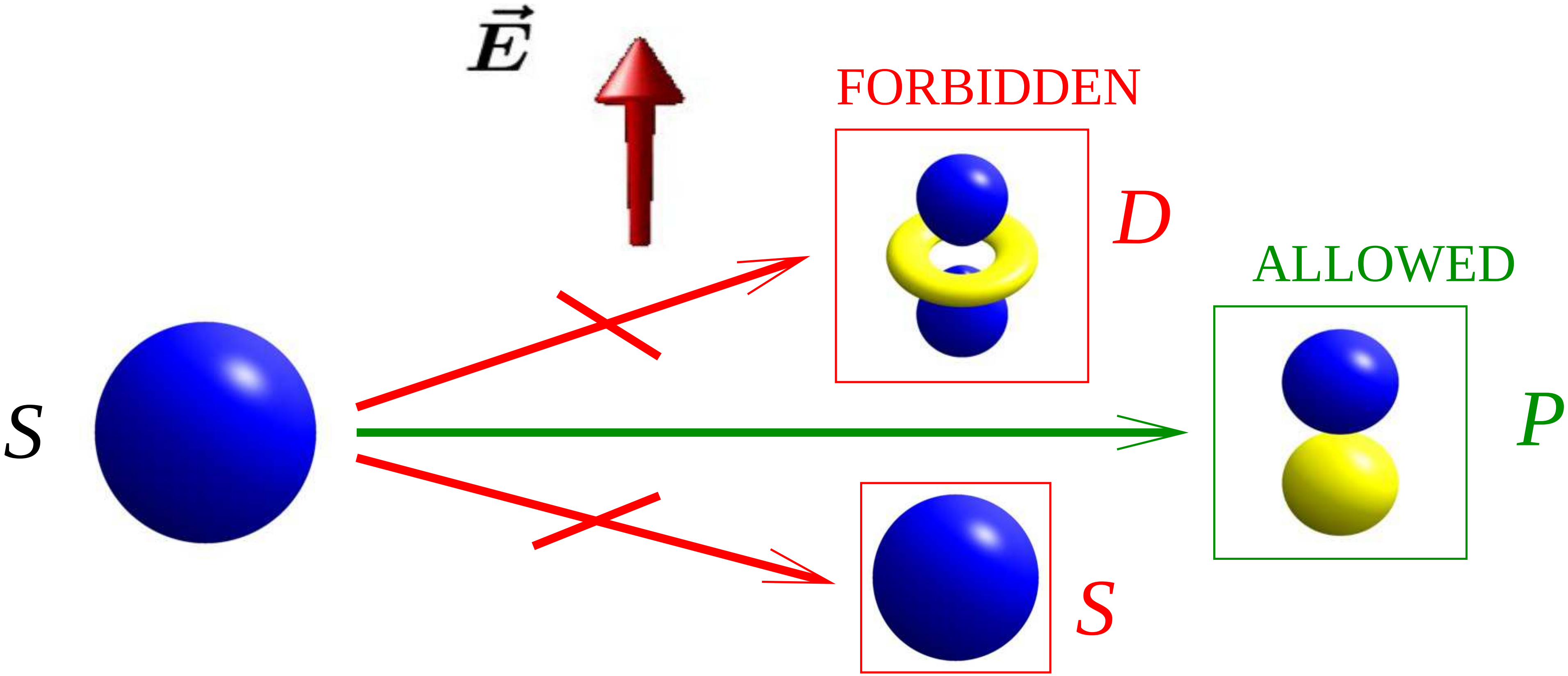

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— the Pauli exclusion principle. WHAT TO SELECT?

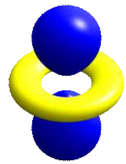
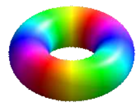
Outline

Permutation-invariant eigenstates for spin-independent interactions:
— $SU(M)$ *vs.* permutation symmetry
— spatially-chaotic states with defined spins
— stability in a presense of spin-dependent interactions

Selection rules, related to the permutation symmetry.

Correlation rules— physical sense of Young diagrams.

MULTIDIMENSIONAL REPRESENTATIONS

Rotation group: $\hat{L}_x$  $(L_z = 1) =$  $(L_z = 0)+$  $(L_z = 2)$

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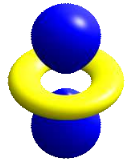
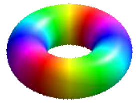
Rotation group: $\hat{L}_x$  $(L_z = 1) =$  $(L_z = 0) +$  $(L_z = 2)$

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Two spin 1/2 atoms, $S_z = 0$

$$|\uparrow\downarrow\rangle, \quad |\downarrow\uparrow\rangle$$

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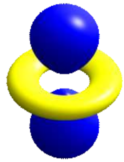
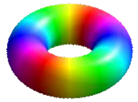
Rotation group: $\hat{L}_x$  $(L_z = 1) =$  $(L_z = 0)+$  $(L_z = 2)$

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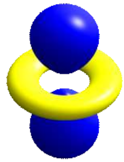
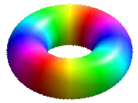
Rotation group: $\hat{L}_x$  $(L_z = 1) =$  $(L_z = 0) +$  $(L_z = 2)$

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$$\Xi_1 = (-|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle)/\sqrt{2}$$

$$\Xi_2 = (2|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle)/\sqrt{3}$$

$$\mathcal{P}\Xi_t = \sum_{t'} D_{t't}(\mathcal{P})\Xi_{t'}$$

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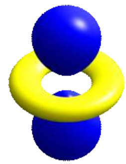
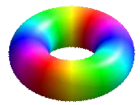
Rotation group: $\hat{L}_x$  $(L_z = 1) =$  $(L_z = 0)+$  $(L_z = 2)$

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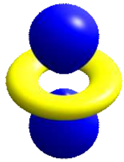
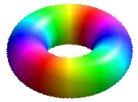
Rotation group: $\hat{L}_x$  $(L_z = 1) =$  $(L_z = 0) +$  $(L_z = 2)$

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SPINOR GASES: SPIN AND SPATIAL DEGREES OF FREEDOM

Experiments: Myatt, Burt, Ghrist, Cornell & Wieman, (1997)
Stamper-Kurn, Andrews, Chikkatur, Inouye, Miesner, Stenger & Ketterle (1998)
Theory: Tin-Lun Ho (1998), Ohmi & Machida (1998)

SPINOR GAS WITH SPIN-INDEPENDENT INTERACTIONS

$$\hat{H} = \hat{H}_{\rm spat} + \hat{H}_{\rm spin}$$

spin-independent $\hat{H}_{\rm spat}$, coordinate-independent $\hat{H}_{\rm spin}$

permutation invariance: $\mathcal{P}^{-1}\hat{H}_{\rm spat}\mathcal{P} = \hat{H}_{\rm spat}$, $\mathcal{P}^{-1}\hat{H}_{\rm spin}\mathcal{P} = \hat{H}_{\rm spin}$

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Spin-free quantum chemistry

1D gases — Yang-Gauden model [Yang (1967), Sutherland (1968)]

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Observation of $SU(M)$ symmetry:

^{87}Sr: $s = 9/2$ — $SU(10)$X. Zhang, M. Bishof, S. L. Bromley, C. V. Kraus, M. S. Safronova, P. Zoller, A. M. Rey, J. Ye, (2014);
^{173}Yb: $s = 5/2$ — $SU(6)$ F. Scazza, C. Hofrichter, M. Höfer, P. C. De Groot, I. Bloch, and S. Fölling, (2014);
G. Pagano, M. Mancini, G. Cappellini, P. Lombardi, Florian Schäfer, Hui Hu, Xia-Ji Liu, J. Catani, C. Sias, M. Inguscio, and L. Fallani (2014).

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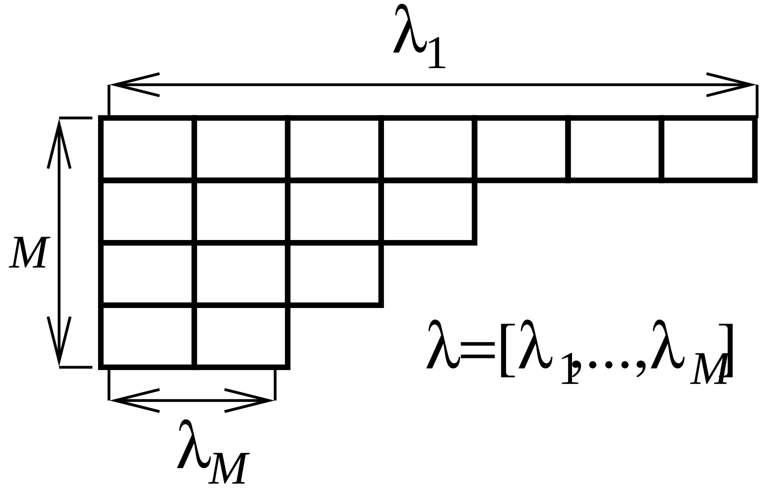

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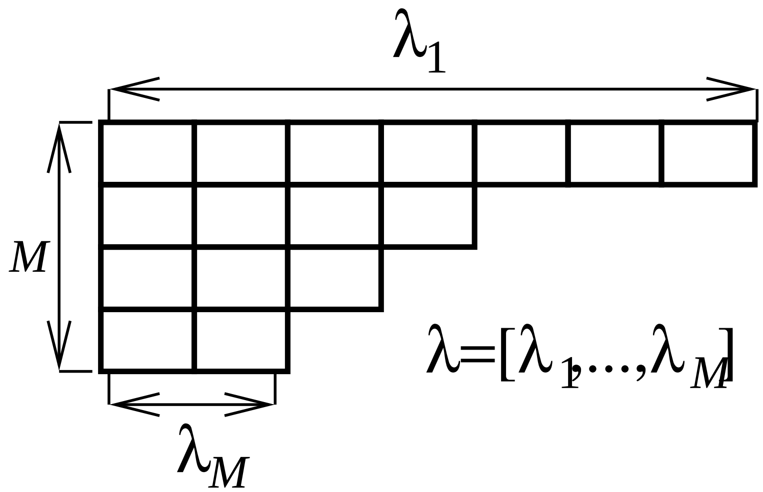

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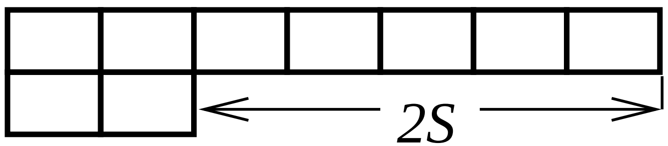
$s = 1/2$: 

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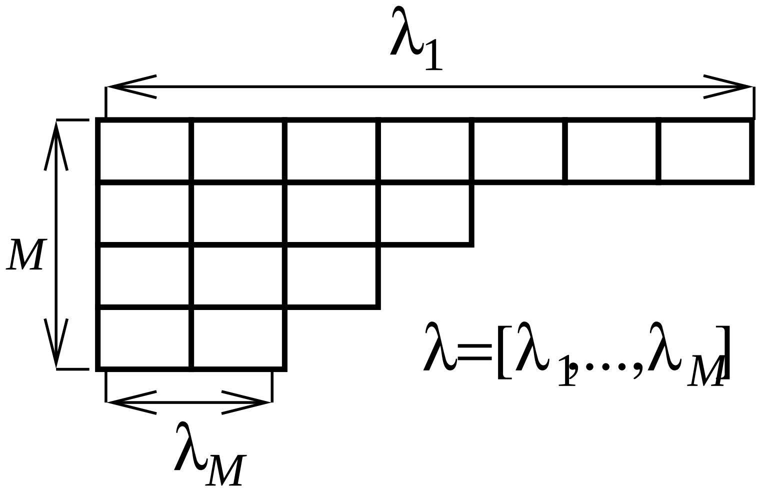

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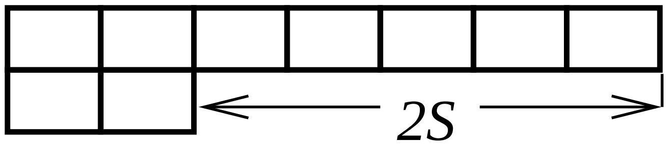
$s = 1/2$: 

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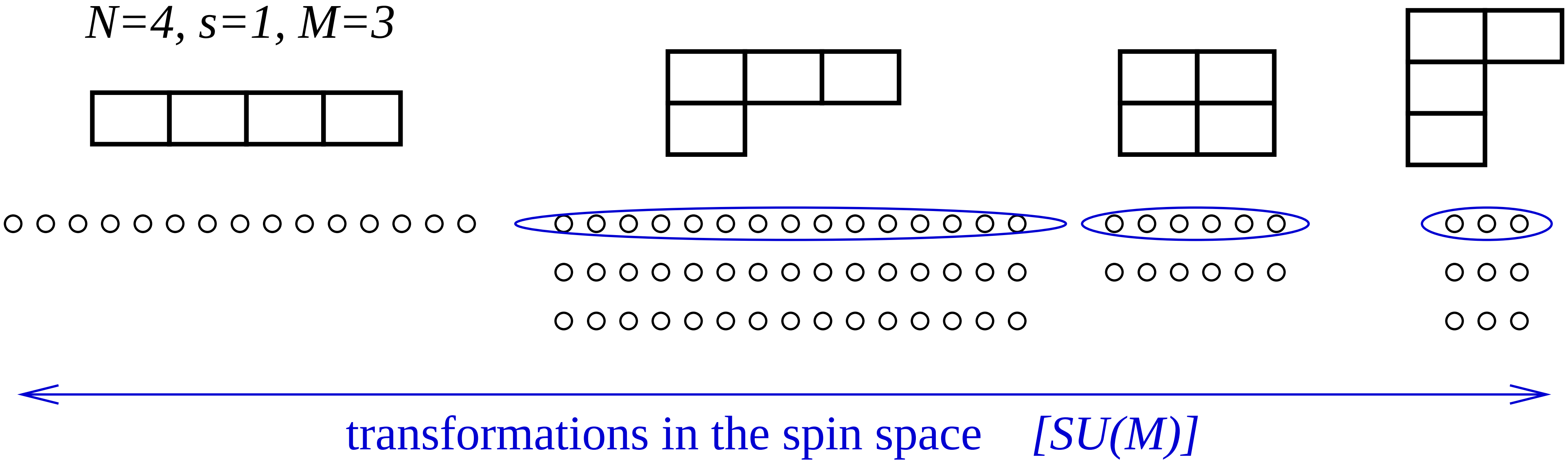
$s > 1/2$: the total spin is undefined, $S_{\max} = (s+1)N - \sum_{m=1}^{M} m\lambda_m$

$SU(M)$ AND PERMUTATION SYMMETRIES



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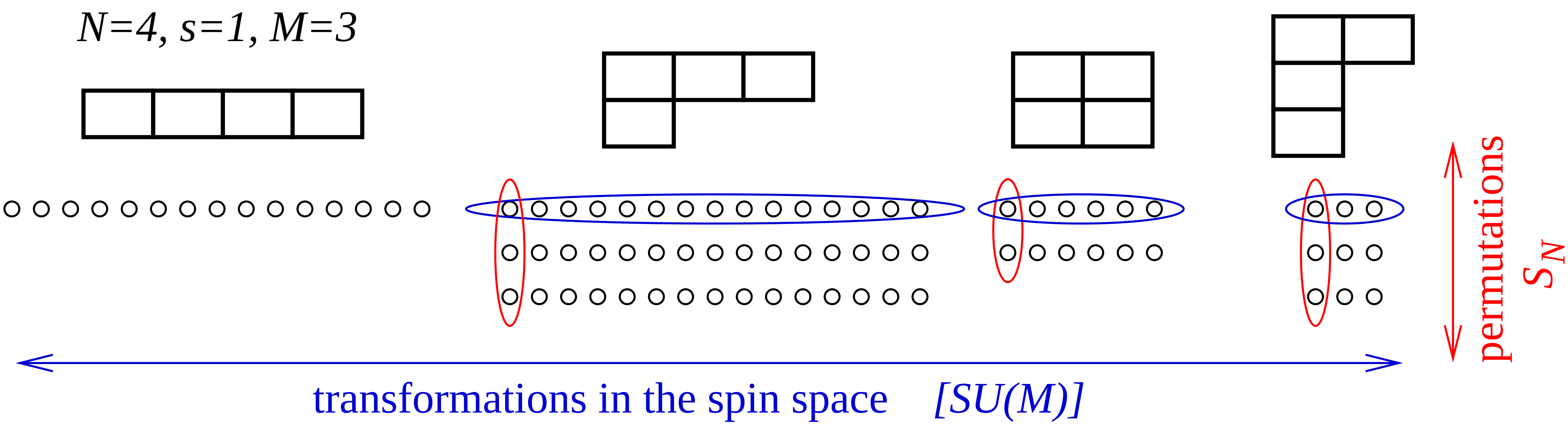

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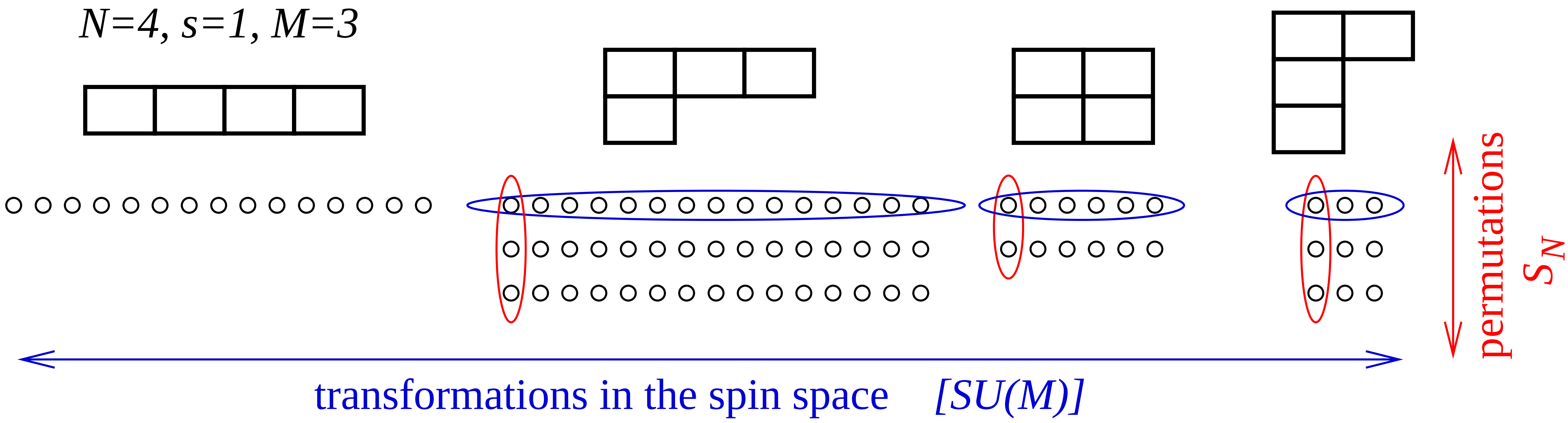


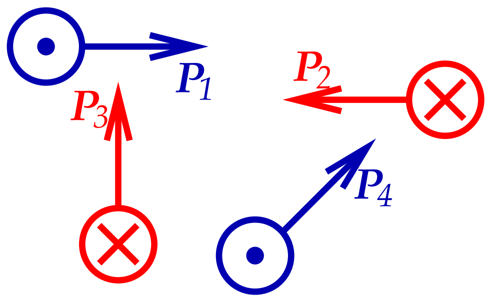
$SU(M)$ — for the spin and total wavefunctions

Permutation symmetry — for the spin and spatial wavefunctions
— can persist if $SU(M)$ is violated
($\hat{H}_{\text{spin}}$ is coordinate-independent but is not $SU(M)$ invariant)

Consequences of the $SU(M)$ and permutation symmetries do not coincide (and do not contradict).

EIGENSTATES OF QUANTUM GASES: INDIVIDUAL SPINS

$$\Psi \propto= \sum_{\mathcal{P}} \prod_j \varphi_{\mathbf{p}_j}(\mathbf{r}_{\mathcal{P}j}) |\sigma_j(\mathcal{P}j)\rangle$$



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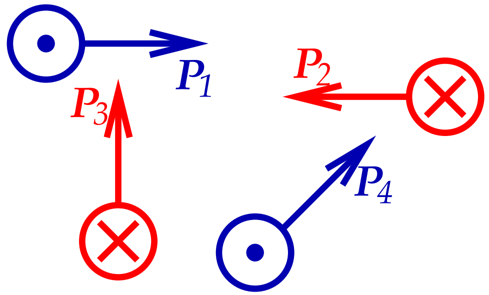
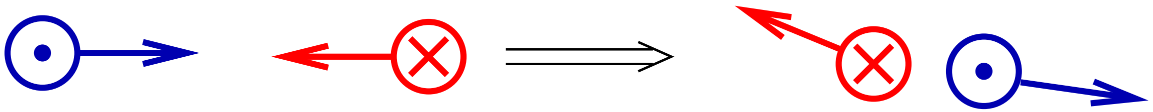


Excited states of interacting gases



Thermal chaotic eigenstate
[Srednicki, PRE 50, 888 (1994)]

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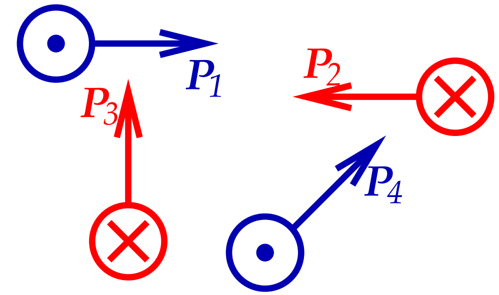

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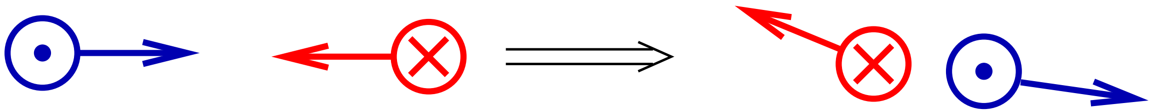

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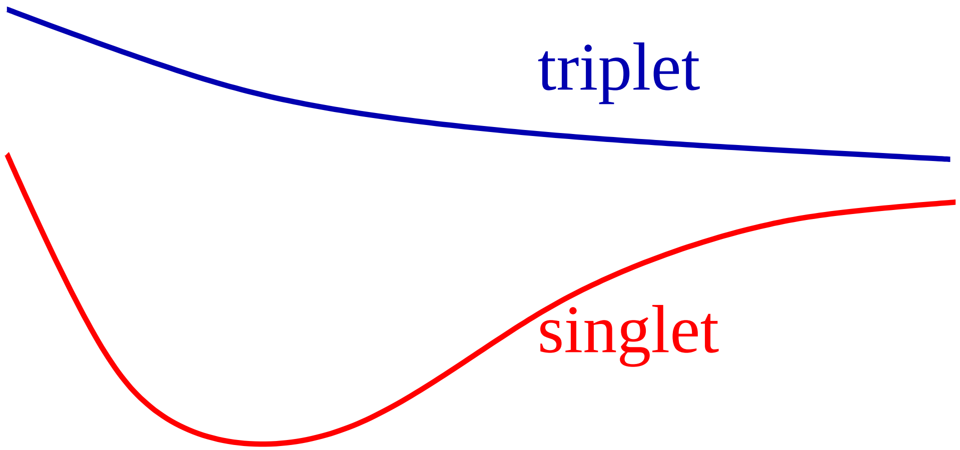
COLLECTIVE SPIN AND SPATIAL WAVEFUNCTIONS



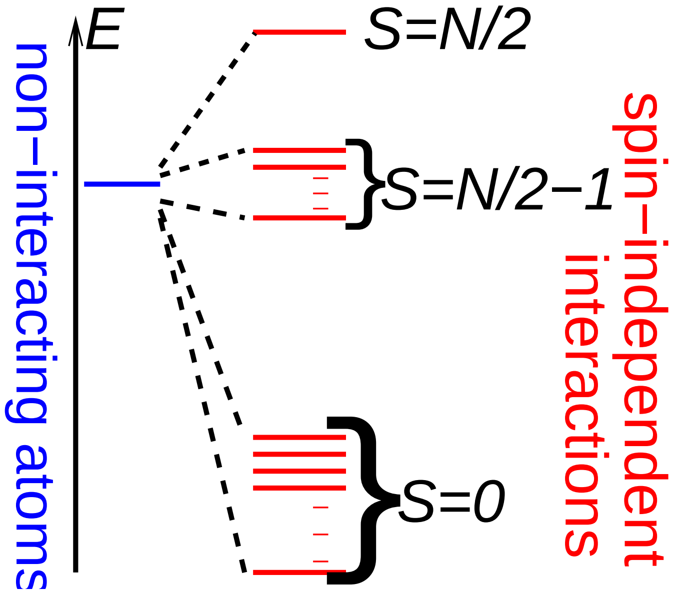
Individual spins of interacting electrons are undefned

MANY-BODY COLLECTIVE SPIN AND SPATIAL WAVEFUNCTIONS

$$s = 1/2$$

Degenerate states with collective spin wavefunctions

Unitary transformation to degenerate states with individual spins



Non-degenerate states with defined total spins S [Heitler, Z. Phys. **46**, 47 (1927)]

S — good quantum number

No thermal equilibrium between states with different S

How long can the states with defined S survive spin-dependent interactions?

SPATIAL CHAOS FOR DEFINED TOTAL SPINS[VY, ArXiv 1509.01264]

Atoms with spin-independent interactions, the total spin S is conserved.

Conditions of chaos — high energy-density of states [Deutch (1991)],

—delocalization in the Fock space

[Altshuler et al (1997),Jacquod & Shepelyansky (1997), Silvestrov (1998)]

$a > 2\pi\hbar/(\sqrt{mT}N^2)$ (3D), $a > 2\sqrt{2\pi\hbar/(m\omega_{conf})}/N^2$ (2D)

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$a > 2\pi\hbar/(\sqrt{mT}N^2)$ (3D), $a > 2\sqrt{2\pi\hbar/(m\omega_{conf})}/N^2$ (2D)
Berry conjecture [Berry (1977)]: an egenfunction is a superposition of plane waves with random phases and Gaussian random amplitudes, but with fixed wavelength.

SPATIAL CHAOS FOR DEFINED TOTAL SPINS[VY, ArXiv 1509.01264]

Atoms with spin-independent interactions, the total spin S is conserved.
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—delocalization in the Fock space
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random phase and amplitude ⇓ ⇓ fixed wavelength

Wavefunctions of interacting atoms $\Psi^{(S)}_{nS_z} = \mathcal{N} \sum_{r,\{\mathbf{p}\}} A^{(S)}_n(r,\{\mathbf{p}\})\delta(\{\mathbf{p}\}^2 - 2mE^{(S)}_n)\,\tilde{\Psi}^{(S)}_{r\{\mathbf{p}\}S_z}$

⇑

wavefunctions of non-interacting atoms — symmetrized plane waves

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$\Uparrow$

wavefunctions of non-interacting atoms — symmetrized plane waves

$$A^{(S)^*}_n(r', \{\mathbf{p}'\}) A^{(S)}_n(r, \{\mathbf{p}\}) = \frac{\delta_{r'r}\delta_{\{\mathbf{p}'\}\{\mathbf{p}\}}}{\delta(\{\mathbf{p}'\}^2 - \{\mathbf{p}\}^2)}$$

(generalization of [Srednicki (1994)])

DECAY OF STATES WITH DEFINED TOTAL SPINS

Weisskopf-Wigner estimate for the decay rate

$$\Gamma = 2\pi |\langle \Psi^{(S')}_{n'S'_z} | \hat{V}_{spin} | \Psi^{(S)}_{nS_z} \rangle|^2 \frac{dn}{dE^{(S')}_{n'}} |_{E^{(S')}_{n'} = E^{(S)}_n}$$

$$|\langle \Psi^{(S')}_{n'S'_z} | \hat{V}_{spin} | \Psi^{(S)}_{nS_z} \rangle|^2 = (\mathcal{N}\mathcal{N}')^2 \sum_{\{\mathbf{p}\}\{\mathbf{p}'\}} \delta(\{\mathbf{p}\}^2 - 2mE^{(S)}_n)\delta(\{\mathbf{p}'\}^2 - 2mE^{(S')}_{n'})$$

calculated with **sum rules** $\longrightarrow$ $\quad \times \sum_{r,r'} |\langle \tilde{\Psi}^{(S')}_{r'\{\mathbf{p}'\}S'_z} | \hat{V}_{spin} | \tilde{\Psi}^{(S)}_{r\{\mathbf{p}\}S_z} \rangle|^2$

[VY, PRA **91**, 053601 (2015); ArXiv 1506.01268]

Spin-dependent two-body interactions

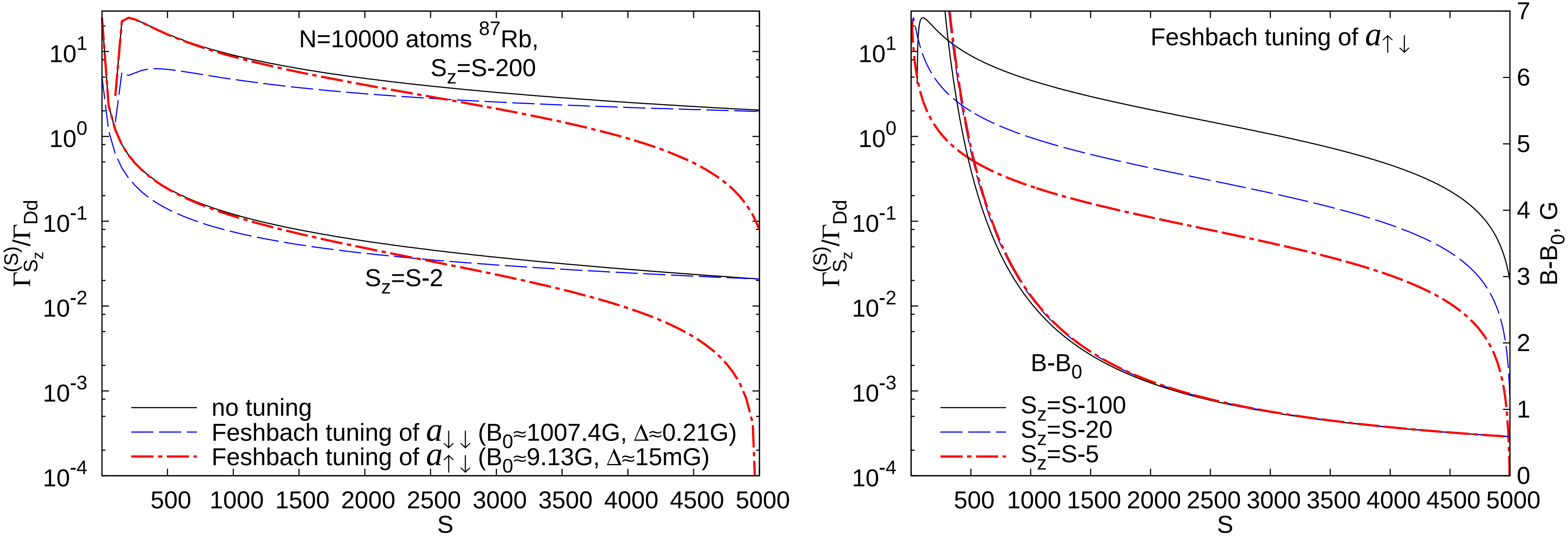
$$\hat{V}_{\rm spin} = \frac{g_{Dd}^{\uparrow\uparrow}}{2}\hat{V}_{\uparrow\uparrow} + \frac{g_{Dd}^{\downarrow\downarrow}}{2}\hat{V}_{\downarrow\downarrow} + g_{Dd}^{\uparrow\downarrow}\hat{V}_{\uparrow\downarrow}, \quad g_{3d} = 4\pi\hbar^2\frac{a}{m}, \quad g_{2d} = g_{3d}\sqrt{\frac{m\omega_{conf}}{2\pi\hbar}}$$

$$\begin{aligned}\Gamma_{S_z}^{(S)} = \Gamma_{Dd}\Bigg\{ & \frac{[(S-1)^2 - S_z^2](S^2 - S_z^2)}{2S(S-1)(2S-1)(2S+1)N}(N+2S)(N+2S+2)\alpha_+^2 \\ & + \frac{(S^2 - S_z^2)(N+2S+2)}{S(2S+1)N}\left[\alpha_+^2\frac{S_z^2(N+2)}{S^2-1} + \alpha_-^2(N-2) + 4\alpha_+\alpha_- S_z\right] \\ & + \frac{[(S+1)^2 - S_z^2](N-2S)}{(S+1)(2S+1)N}\left[\alpha_+^2\frac{S_z^2(N+2)}{S(S+2)} + \alpha_-^2(N-2) + 4\alpha_+\alpha_- S_z\right] \\ & + \frac{[(S+1)^2 - S_z^2][(S+2)^2 - S_z^2]}{2(S+1)(S+2)(2S+1)(2S+3)N}(N-2S)(N-2S-2)\alpha_+^2\Bigg\}\end{aligned}$$

$$\alpha_+ = (a_{\uparrow\uparrow} + a_{\downarrow\downarrow} - 2a_{\uparrow\downarrow})/a\ ,\alpha_- = (a_{\uparrow\uparrow} - a_{\downarrow\downarrow})/a,\ a = (a_{\uparrow\uparrow} + a_{\downarrow\downarrow} + a_{\uparrow\downarrow})/3$$

$$\Gamma_{3d} = 2\sqrt{\pi T/m}a^2 n_{3d}, \quad \Gamma_{2d} = \frac{\pi}{2}a^2\omega_{conf}n_{2d}$$

THE DECAY MINIMIZATION BY FESHBACH RESONANCE



^{87}Rb, $n_{3d} = 10^{12}\text{cm}^{-3}$, $T = 1\mu\text{K}$: $\Gamma_{3d} \approx 0.9\text{s}^{-1}$

$a_{\uparrow\downarrow}$ tuning at $S_z \approx \pm S$, $N \gg 1$: $\alpha_+ \approx \mp\alpha_-$

$$\Gamma^{(S)}_{S_z} \sim \frac{\alpha_+^2 \Gamma_{Dd}}{N}$$

TWO-BODY CORRELATIONS

Local correlations of atoms in different spin-states
— calculated with sum rules.

Averaged over states with defined total spin

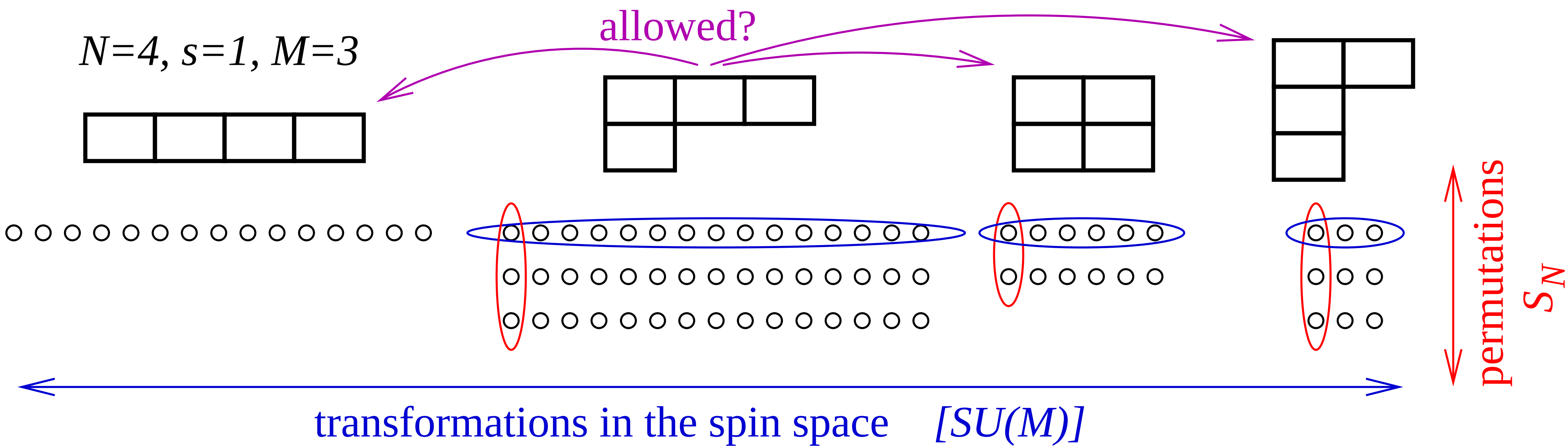
$$\bar{\rho}_{\uparrow\downarrow}^{(S,S_z)} = \frac{\langle \rho_2(0) \rangle}{N(N-1)} \left(\frac{1}{4} N(N-2) + S(S+1) - 2S_z^2 \right)$$

$\langle \rho_2(0) \rangle$ — the average two-body density, independent of S and S_z

Averaged over the thermal equilibrium with given total spin projection (occupations of the individual spin-states)

$$\bar{\rho}_{\uparrow\downarrow}^{(N_\uparrow,N_\downarrow)} = \frac{\langle \rho_2(0) \rangle}{N(N-1)} \left(\frac{1}{4} N^2 - S_z^2 \right) = \bar{\rho}_{\uparrow\downarrow}^{(S,S_z)} - \frac{\langle \rho_2(0) \rangle}{N(N-1)} \left(S^2 - S_z^2 + S - \frac{N}{4} \right)$$

SELECTION RULES (arbitrary s)



$$\hat{H} = \hat{H}_{\rm spat} + \hat{H}_{\rm spin} + \sum_{1\le j_1<j_2<\cdots<j_k\le N} \hat{W}_k(j_1,\ldots,j_k)$$

$$\hat{W}_k(\{j\}) = \sum_{\{m\},\{m'\}} \langle\{m'\}|\hat{W}(\mathbf{r}_{j_1},\ldots,\mathbf{r}_{j_k})|\{m\}\rangle \prod_{i=1}^{k} |m'_i(j_i)\rangle\langle m_i(j_i)|,$$

— k-body interaction, depends on the spin states $|m(j)>$ and coordinates $\mathbf{r}_j$ of k atoms $j_1,\ldots,j_k$

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$$\langle \Psi^{[\lambda']} | \hat{W}_k(\{j\}) | \Psi^{[\lambda]} \rangle = \langle \Psi^{[\lambda']} | \hat{W}_k(N-k, \ldots, N) | \Psi^{[\lambda]} \rangle$$

(both $\Psi^{[\lambda]}$ and $\Psi^{[\lambda']}$ are either symmetric or antisymmetric)
Invariance over permutations of the first $N-k$ atoms
— reduction to subgroup $S_N \Rightarrow S_{N-k}$
Orthogonality relations.

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λ and λ' differ by relocation of no more than k boxes between their rows

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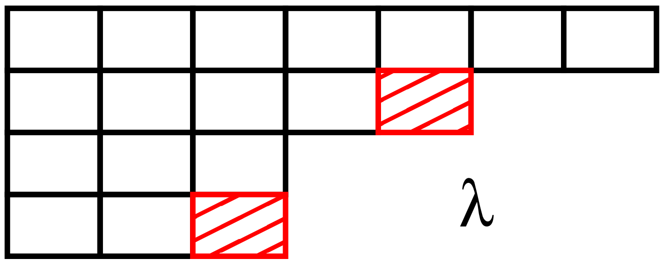
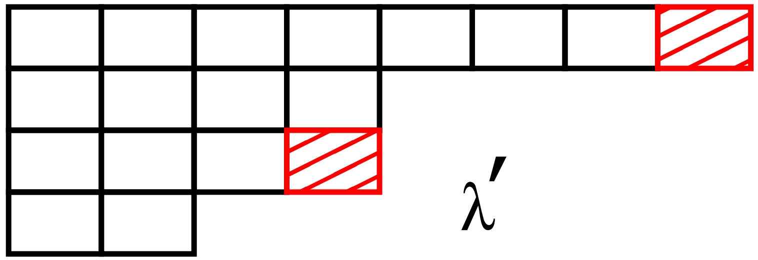
Two-body interaction ($k = 2$)



λ



λ′

SELECTION RULES

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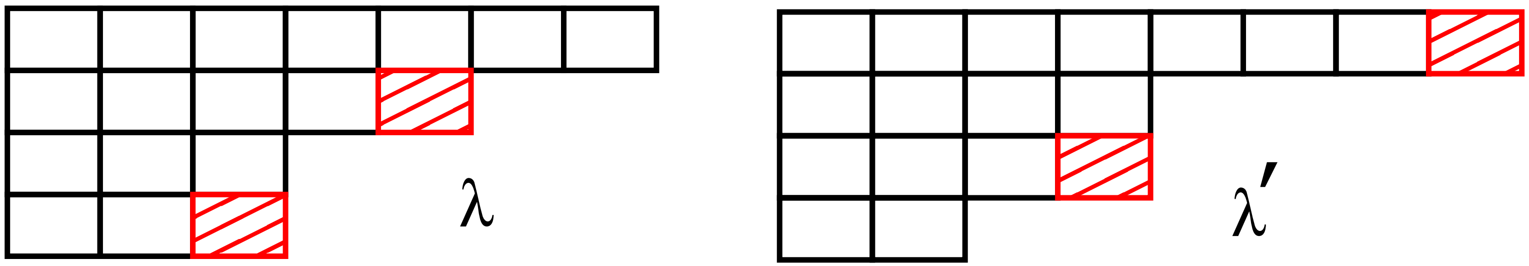
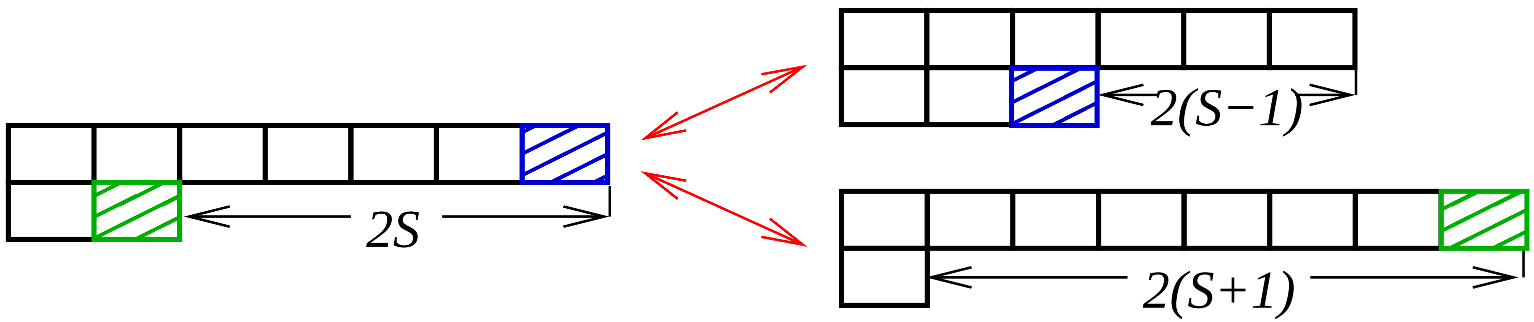


$$\sum_{m=1}^{M} |\lambda_m - \lambda_m'| \leq 2k$$

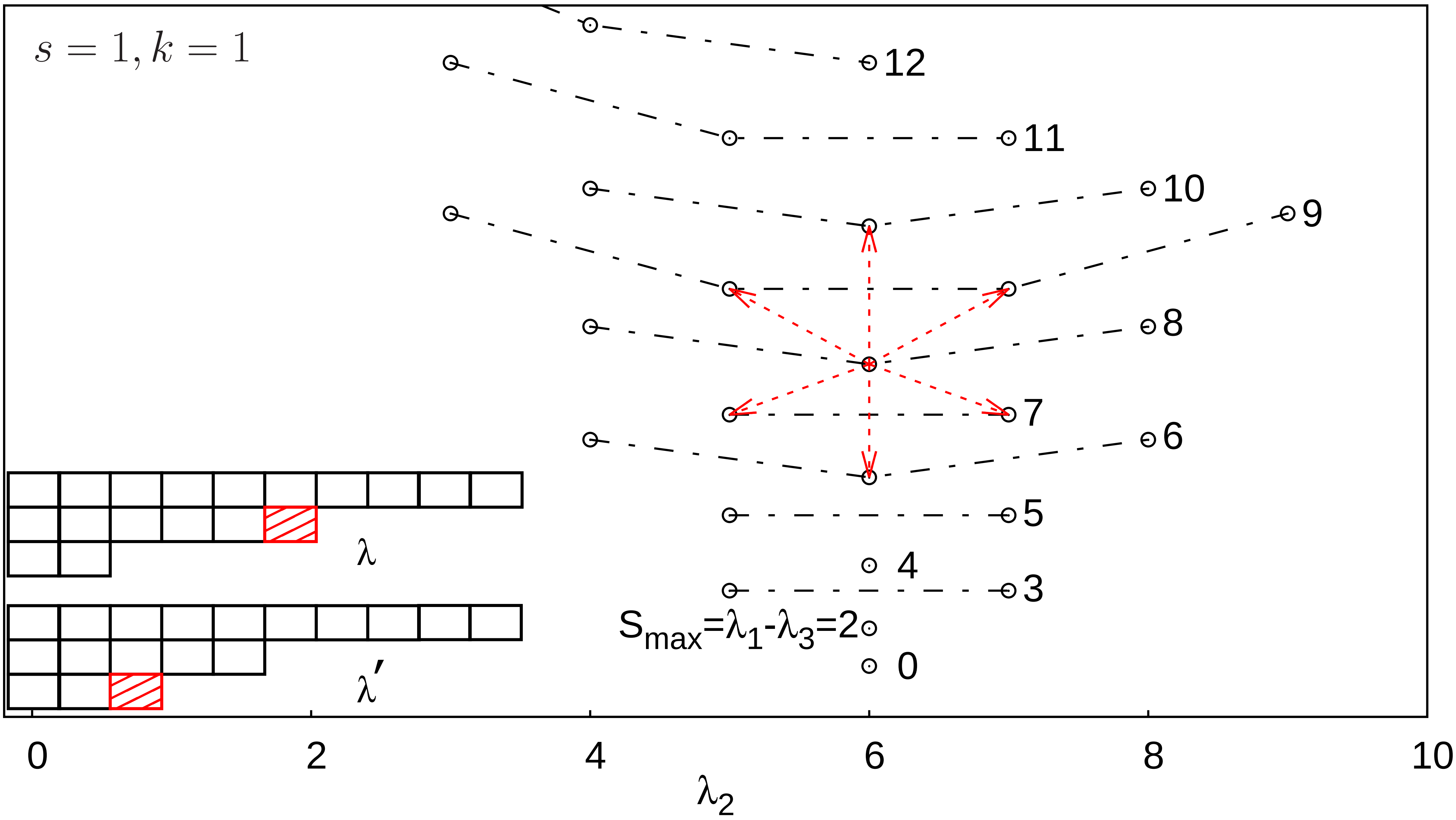
DIPOLE TRANSITIONS

$$s = 1/2, k = 1$$



$$\hat{W}_k = C_0 + C_x \hat{S}_x + C_y \hat{S}_y + C_z \hat{S}_z$$

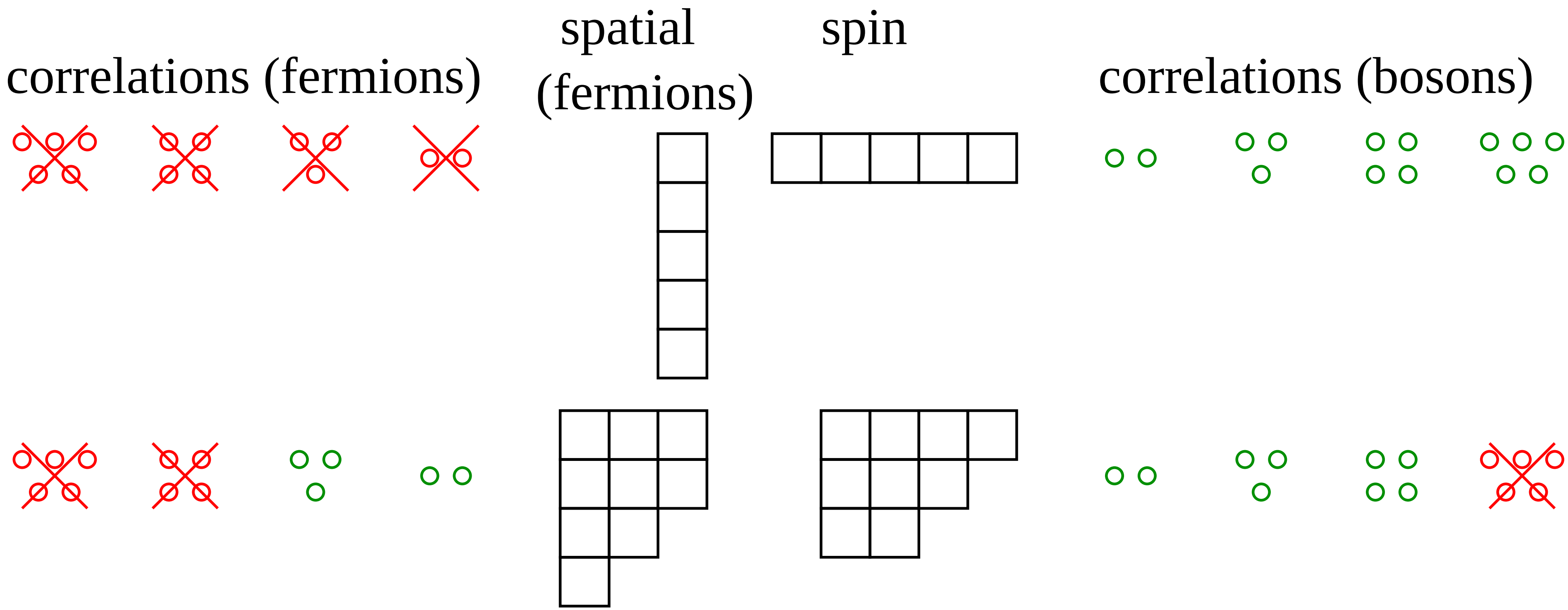
HIGH SPIN



$s = 1, k = 1$
12
11
10
9
8
7
6
5
4
3
0
λ
λ′
$S_{max}=\lambda_1-\lambda_3=2$
0
2
4
6
8
10
λ_2

CORRELATION RULES

Eigenstate-averaged local correlations of k particles vanish if k exceeds the number of columns (for bosons) or rows (for fermions) in the associated Young diagram



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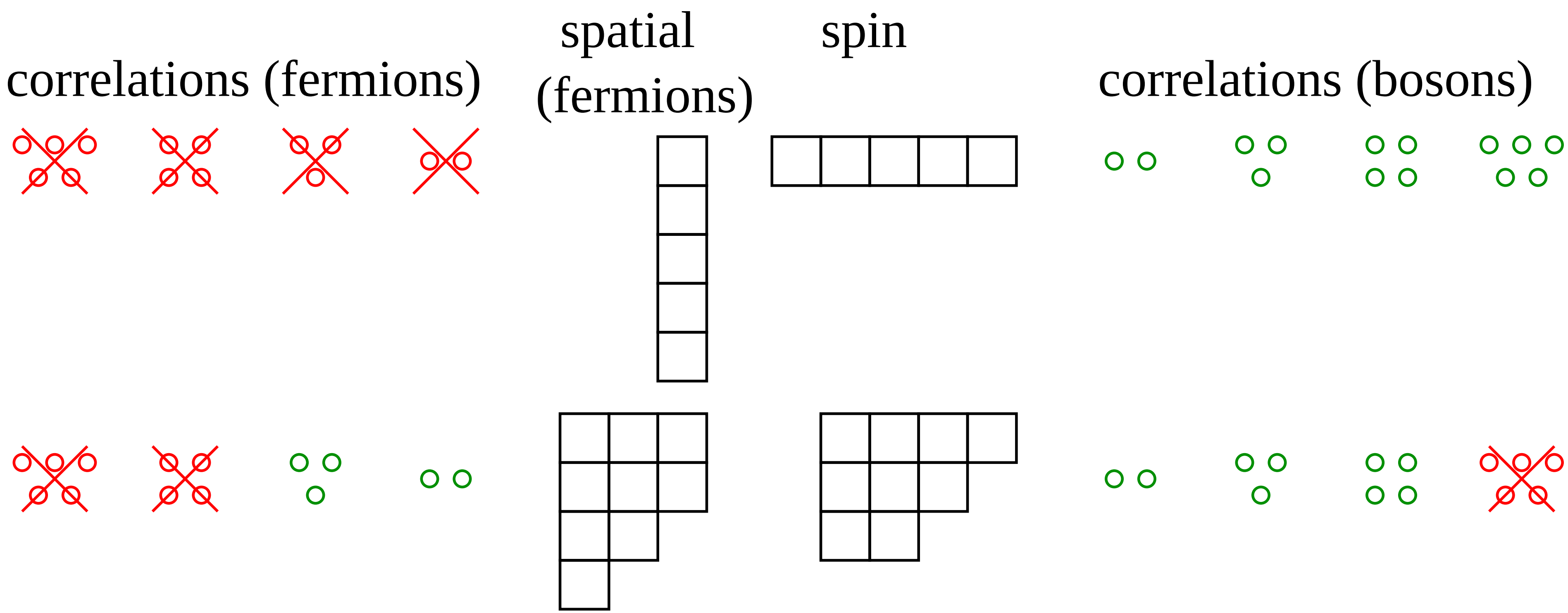

Valid for both spatial $\langle\Psi^{[\lambda]}|\prod_{i=2}^{k}\delta(\mathbf{r}_1-\mathbf{r}_i)|\Psi^{[\lambda]}\rangle$
and momentum $\langle\Psi^{[\lambda]}|\prod_{i=2}^{k}\delta(\mathbf{p}_1-\mathbf{p}_i)|\Psi^{[\lambda]}\rangle$ correlations

POPULATION OF MANY-BODY STATES

Spatially-homogeneous spin-changing (conserves the Young diagram)

$\hat{W}_{\mathrm{hom}}(t) = \sum_{m \neq m'} W_{mm'}(t) |m\rangle\langle m'|$ (example — the $\pi/2$ pulse)

Spatially-inhomogeneous spin-conserving (moves one box)

$\hat{W}_{\mathrm{inh}}(\mathbf{r}, t) = \sum_{m} W_{m}(\mathbf{r}, t) |m\rangle\langle m|$

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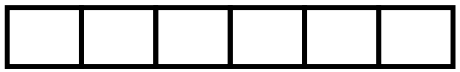
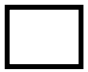




1 spin state

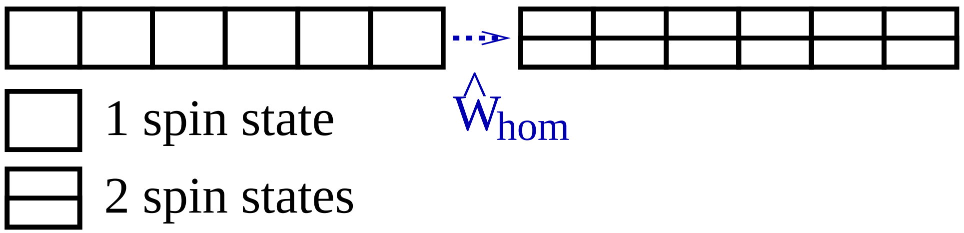
POPULATION OF MANY-BODY STATES

Spatially-homogeneous spin-changing (conserves the Young diagram)

$\hat{W}_{\rm hom}(t) = \sum_{m \neq m'} W_{mm'}(t)|m\rangle\langle m'|$ (example — the $\pi/2$ pulse)

Spatially-inhomogeneous spin-conserving (moves one box)

$\hat{W}_{\rm inh}(\mathbf{r},t) = \sum_{m} W_{m}(\mathbf{r},t)|m\rangle\langle m|$



fermion correlations 

POPULATION OF MANY-BODY STATES

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$\hat{W}_{\mathrm{hom}}(t) = \sum_{m \neq m'} W_{mm'}(t) |m\rangle\langle m'|$ (example — the $\pi/2$ pulse)

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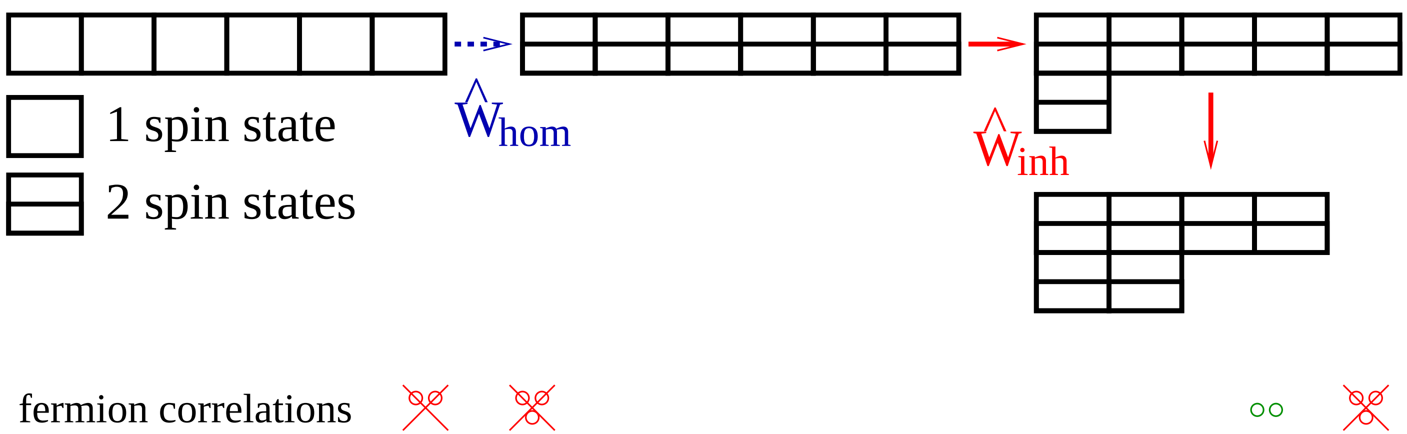

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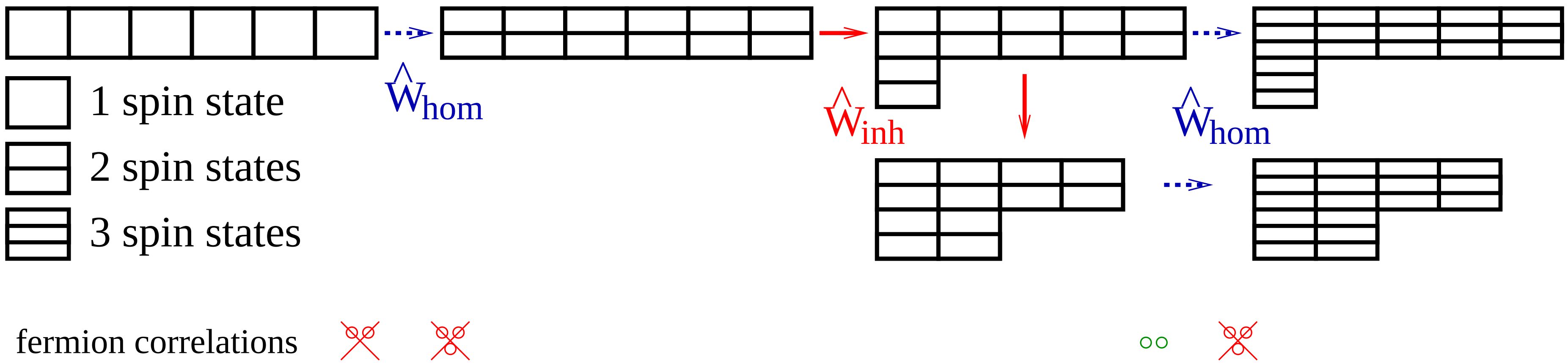

CONSERVATION LAWS

Integrals of moton — **the character operators**
[Dirac, Proc. R. Soc. A (1929)]

$$\hat{\chi}(C_N) = \sum_{\mathcal{P}\in C_N} \mathcal{P}/g(C_N)$$

(The sum is over all $g(C_N)$ permutations in a conjugate class C_N)
Their **eigenvalues** $\tilde{\chi}_\lambda(C_N)$ — the normalized characters

Each spatial orbital is occupied only by one particle

Average spatial and momentum local correlations
$\bar{\rho}_k^{[\lambda]}(\{0\}) = \tilde{\rho}_k^{[\lambda]}\langle\rho_k(\{0\})\rangle$, $\bar{g}_k^{[\lambda]}(\{0\}) = \tilde{\rho}_k^{[\lambda]}\langle g_k(\{0\})\rangle$
$\langle\rho_k(\{0\})\rangle$ and $\langle g_k(\{0\})\rangle$ λ— multiplet independent
Multiplet dependence — the universal factor

$$\tilde{\rho}_k^{[\lambda]} = \sum_{C_k} \mathrm{sig}(C_k)g(C_k)\tilde{\chi}_\lambda(C_k)$$