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Stable two-dimensional composite solitons in spin-orbit-coupled self-attractive Bose-Einstein condensates in free space

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Creation of two-dimensional composite solitons in spin-orbit-coupled self-attractive Bose-Einstein condensates in free space

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It is commonly known that two-dimensional mean-field models of optical and matter waves with cubic self-attraction cannot produce stable solitons in free space because of the occurrence of collapse in the same setting. By means of numerical analysis and variational approximation, we demonstrate that the two-component model of the Bose-Einstein condensate with the spin-orbit Rashba coupling and cubic attractive interactions gives rise to solitary-vortex complexes of two types: semivortices (SVs, with a vortex in one component and a fundamental soliton in the other), and mixed modes (MMs, with topological charges 0 and ± 1 mixed in both components). These two-dimensional composite modes can be created using the trapping harmonic-oscillator (HO) potential, but remain stable in free space, if the trap is gradually removed. The SVs and MMs realize the ground state of the system, provided that the self-attraction in the two components is, respectively, stronger or weaker than the cross attraction between them. The SVs and MMs which are not the ground states are subject to a drift instability. In free space (in the absence of the HO trap), modes of both types degenerate into unstable Townes solitons when their norms attain the respective critical values, while there is no lower existence threshold for the stable modes. Moving free-space stable solitons are also found in the present non-Galilean-invariant system, up to a critical velocity. Collisions between two moving solitons lead to their merger into a single one.

(1) Introduction and objectives

Recently, the concept of ***emulation*** (alias ***simulation***) of complex physical effects, known in condensed-matter physics, by much simpler settings available in **BEC** (***matter waves***) and **photonics** (***optical waves***), has drawn a great deal of interest:

P. Hauke, F. M. Cucchietti, L. Tagliacozzo, I. Deutsch, and M. Lewenstein, Rep. Prog. Phys. **75**, 082401 (2012).

Lately, a new topic has emerged in the framework of this approach: the emulation of **spin-orbit** (**SO**) **interactions** in semiconductors, such as those accounted for by the ***Rashba*** and ***Dresselhaus*** terms, by mapping the spinor wave function of electrons into the pseudo-spinor two-component wave function of a binary **BEC** gas:

Y. J. Lin, K. Jimenez-Garcia, and I. B. Spielman, Nature **471**, 83 (2011);

Y. Zhang, L. Mao, and C. Zhang, Phys. Rev. Lett. **108**, 035302 (2012);

A brief review: *H. Zhai*, Int. J. Mod. Phys. B **26**, 1230001 (2012).

The **SO** coupling is a ***linear feature*** of the system. Its combination with the natural ***cubic nonlinearity*** of the atomic **BEC** gives rise to new effects, such as ***vortices***:

C. J. Wu, Mod. Phys. Lett. B **23**, 1 (2009);
X.-Q. Xu and J. H. Han, Phys. Rev. Lett. **107**, 200401 (2011);
J. Radic', T. A. Sedrakyan, I. B. Spielman, and V. Galitski, Phys. Rev. A **84**, 063604 (2011);
X.-J. Liu, H. Pu, P. D. Drummond, and H. Hu, *ibid*. **85**, 023606 (2012);
H. Sakaguchi and B. Li, *ibid*. **87**, 015602 (2013);

A. Fetter, *ibid*. **89**, 023629 (2014).

Solitons have also been predicted in nonlinear **SO-coupled** systems:

Y. Xu, Y. Zhang, and B. Wu, Phys. Rev. A **87**, 013614 (2013);

L. Salasnich and B. A. Malomed, *ibid*. **87**, 063625 (2013);

V. Achilleos, D. J. Frantzeskakis, P. G. Kevrekidis, and D. E. Pelinovsky, Phys. Rev. Lett. **110**, 264101 (2013);

Y. V. Kartashov, V. V. Konotop, and F. Kh. Abdullaev, *ibid*. **111**, 060402 (2013).

The systems of coupled ***Gross-Pitaevskii equations*** (**GPE**s) for the pseudo-spinor binary **BEC**s not only may emulate the condensed-matter settings, but they may also be tantamount to coupled systems of nonlinear Schrödinger equations (**NLSE**s) describing the co-propagation of ***two polarizations of light*** in ***bimodal*** optical media.

In particular, the pseudo-spinor **SO-coupled GPE** system, which was considered in *V. Achilleos, D. J. Frantzeskakis, P. G. Kevrekidis, and D. E. Pelinovsky*, Phys. Rev. Lett. **110**, 264101 (2013), is identical to the **NLSE** system for two polarizations of light in a ***twisted*** nonlinear optical fiber, *B. A. Malomed*, Phys. Rev. A **43**, 410 (1991).

Getting back to **2D vortices**, they have been studied in detail in models of the **SO-coupled BEC** trapped in the ***harmonic-oscillator*** (**HO**) potential, under the action of the ***intra-species*** (alias **SPM**) and ***inter-species*** (alias **XPM**) ***repulsive nonlinearities***.

Our objective is to construct ***self-trapped*** (localized) ***stable*** **2D** ***vortical modes*** in the **SO-coupled BEC** with ***attractive*** **SPM** and **XPM** nonlinearities, in the ***free space*** (without any trapping potential)

At the first glance, this objective seems ***absolutely impossible***. Formal **2D** ***vortex-soliton*** solutions (alias "***vortex Townes' solitons***") of the **NLSE** with the ***self-attractive*** cubic term are well known:

V. I. Kruglov, Y. A. Logvin, and V. M. Volkov, J. Mod. Opt. **39**, 2277 (1992).

However, it is commonly known too that such solitons are subject to ***strong instability***, against ***splitting*** and ***collapse***. ***Fundamental*** (zero-vorticity) **2D** solitons (***Townes' solitons*** proper) are unstable too, against the ***collapse***.

2D solitons described by the conventional GPE with the attractive cubic nonlinearity can be readily ***stabilized*** (***fully*** stabilized for fundamental solitons, and ***partly*** for vortical ones) by a trapping **HO** potential:

F. Dalfovo and S. Stringari, Phys. Rev. A **53**, 2477 (1996);
R.J. Dodd, J. Res. Natl. Inst. Stand. Technol. **101**, 545 (1996);
T.J. Alexander and L. Berge', Phys. Rev. E **65**, 026611 (2002);
L. D. Carr and C. W. Clark, Phys. Rev. Lett. **97**, 010403 (2006);
G. Herring, L.D. Carr, R. Carretero-Gonzalez, P.G. Kevrekidis, and D.J. Frantzeskakis, *ibid*. **77**, 023625 (2008);
M. Brtka, A. Gammal, and B. A. Malomed, *ibid*. **82**, 053610 (2010).

In the free space, families of **2D** Townes' solitons supported by the ***cubic attractive nonlinearity*** are ***degenerate***: the norm of each family (fundamental, vortical, etc.) takes a ***single value***, which does not depend on the soliton's ***chemical potential*** (alias ***propagation constant***, in optics). This unique value is the one which separates ***collapsing*** and ***decaying*** localized modes. The ***stabilizing mechanism***, provided by the **HO** potential, acts by letting the solitons' norm take values ***below*** this critical value, hence they cannot undergo the ***collapse***, and thus become ***stable*** against small perturbations.

We demonstrate here that the interplay of the linear **SO coupling** (of the ***Rashba*** type) between the two components of the **BEC** pseudo-spinor wave function and the combination of the **SPM** and **XPM** ***self-attractive*** cubic terms gives rise to families of ***composite*** two-component solitons (***half-fundamental – half-vortical***) in the ***free space***, whose norm falls ***below the critical value***. This fact ***stabilizes*** the ***composite solitons*** against the ***collapse***, actually making them ***fully stable*** against small perturbations, ***without the help*** of any trapping potential.

The rest of the talk is organized as follows:

(2) The model will be introduced (the system of coupled **GPEs** in **2D**).

(3) Composite solitons in the form of ***semi-vortices*** will be presented.

(4) Another family of composite solitons, ***mixed modes***, will be presented too.

(5) ***Excited states*** of the ***semi-vortices*** and ***mixed states*** will be added to the picture (although, they are always unstable).

(5) ***Moving solitons*** and ***collisions*** between.

(6) Conclusions.

(2) The model

The system of **GPES** for the (pseudo-) spinor wave function **(Φ_{+}, Φ_{-})** of the binary **BEC** coupled by the **SO** terms of the ***Rashba type*** with strength **$\lambda \equiv 1$**, coefficient of the **SPM** self-attraction **≡ 1**, coefficient of the **XPM** inter-component attraction **$\gamma \geq 0$**, and (***for the time being***) the strength of the **HO** trapping potential **Ω**:

$$i\frac{\partial \phi_+}{\partial t} = -\frac{1}{2}\nabla^2\phi_+ - (|\phi_+|^2 + \gamma|\phi_-|^2)\phi_+ + \lambda\left(\frac{\partial \phi_-}{\partial x} - i\frac{\partial \phi_-}{\partial y}\right) + \frac{1}{2}\Omega^2(x^2+y^2)\phi_+,$$

$$i\frac{\partial \phi_-}{\partial t} = -\frac{1}{2}\nabla^2\phi_- - (|\phi_-|^2 + \gamma|\phi_+|^2)\phi_- - \lambda\left(\frac{\partial \phi_+}{\partial x} + i\frac{\partial \phi_+}{\partial y}\right) + \frac{1}{2}\Omega^2(x^2+y^2)\phi_-,$$

(3) Semi-vortex states

The coupled GPEs admit a family of solutions for ***semi-vortices***, with vorticities ***m*₊ = 0** in one component, and ***m*₋ = 1** in the other. The ***exact ansatz*** for these solutions is

$$\phi_+\left(x,y,t\right)=e^{-i\mu t}f_1(r^2),\phi_-\left(x,y,t\right)=e^{-i\mu t+i\theta}rf_2(r^2),$$

where μ is the chemical potential, $\left(r,\theta\right)$ are the polar

coordinates, and real functions $f_{1,2}(r^2)$ obey the following equations:

$$\mu f_1+2\left[r^2\frac{d^2f_1}{d(r^2)^2}+\frac{df_1}{d(r^2)}\right]+\left(f_1^2+\gamma r^2f_2^2\right)f_1 - 2\lambda\left[r^2\frac{df_2}{d(r^2)}+f_2\right]-\frac{\Omega^2}{2}r^2f_1=0,$$

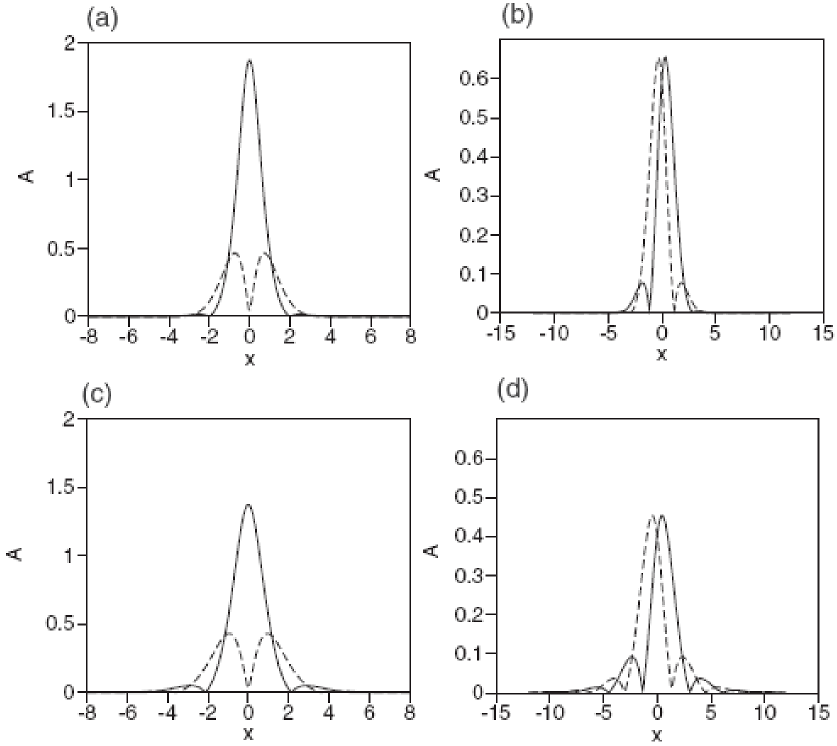
$$\mu f_2+2\left[r^2\frac{d^2f_2}{d(r^2)^2}+2\frac{df_1}{d(r^2)}\right]+\left(r^2f_2^2+\gamma f_1^2\right)f_2 + 2\lambda\frac{df_1}{d(r^2)}-\frac{\Omega^2}{2}r^2f_2=0.$$

The asymptotic form of the amplitude functions, f_1 and f_2, in the **free space** ($\Omega = 0$) at $r \to \infty$:

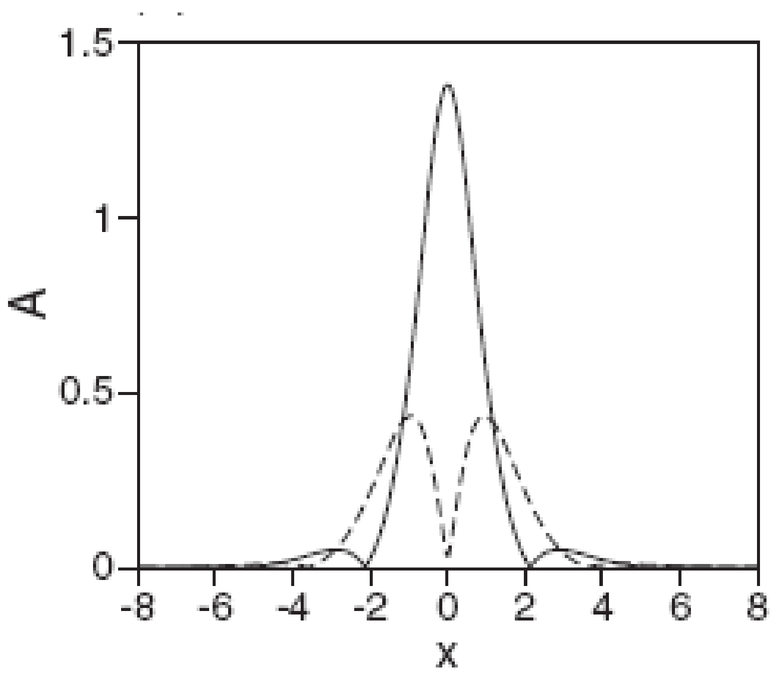
$$f_1(r) \approx F r^{-1/2} \exp\left(-\sqrt{-2\mu - \lambda^2}\, r\right) \cos(\lambda r + \delta), \; f_2(r) \approx F r^{-1/2} \exp\left(-\sqrt{-2\mu - \lambda^2}\, r\right) \sin(\lambda r + \delta),$$

hence such **localized modes** exist for chemical potential $\mu < -\lambda^2 / 2$.

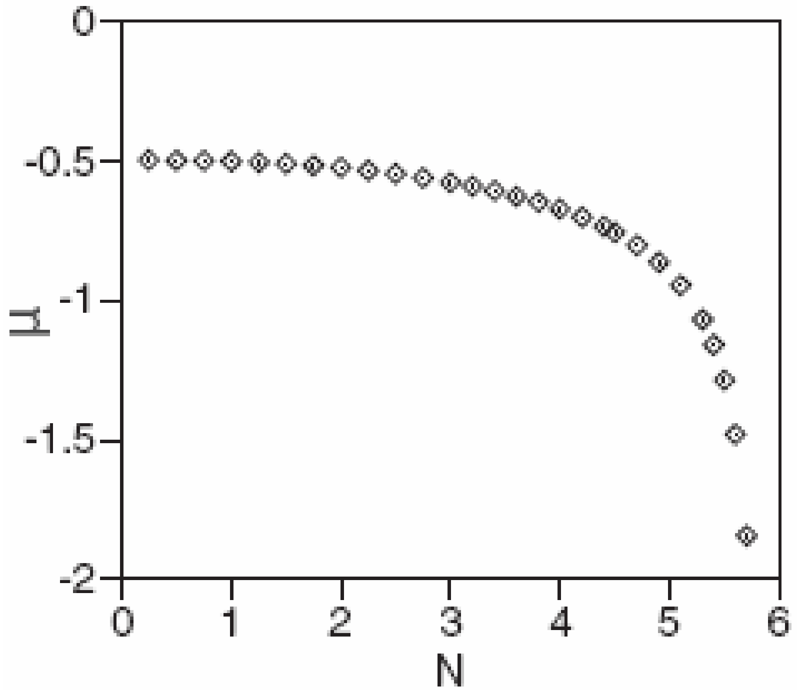
Numerically found cross-section profiles of the 2D semi-vortices (along $y = 0$) for $\Omega = 0.5$ and $\gamma = 0$ **(a)** or $\gamma = 2$ **(b)**, and profiles of counterparts of these modes in the **free space**, **(c)** and **(d)**, produced by **adiabatically reducing** Ω from 0.5 to 0.



The same semi-vortex as above, with ***γ* = 0** and **Ω = 0**, but obtained ***directly*** (by means of the imaginary-time integration) as a stationary solution in the ***free space***:

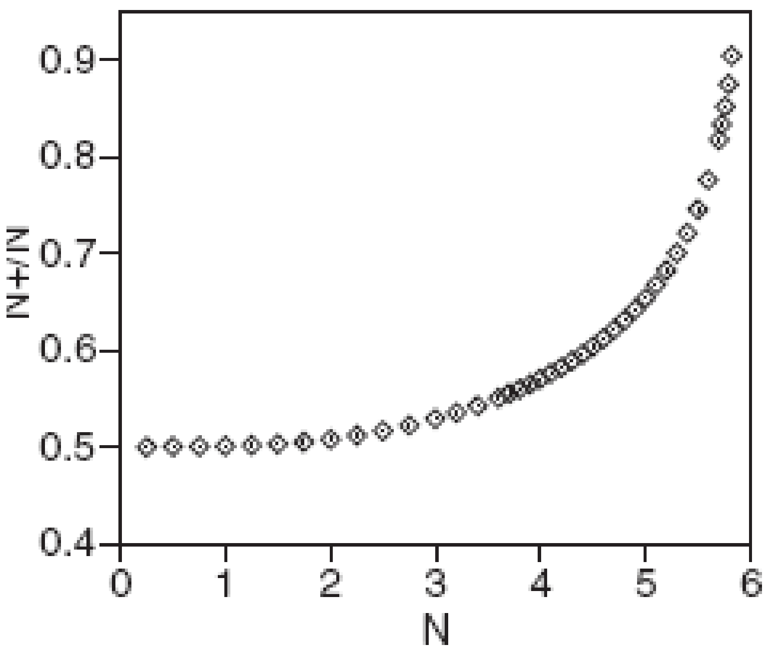


The numerically found dependence between the total norm of the semi-vortices and their chemical potential demonstrates that **(1)** the norm of the semi-vortex indeed falls ***below the critical value***; **(2)** the dependence satisfies the ***Vakhitov-Kolokolov*** **(VK)** ***criterion***, $d\mu/dN < 0$, which is a ***necessary condition*** for the stability; **(3)** there is ***no finite minimum (threshold) value*** of the norm necessary for the existence of the semi-vortex; **(4)** the norm is ***bounded from above*** precisely by the ***critical value***.



In agreement with the **VK criterion**, direct simulations demonstrate that the semi-vortices are ***completely stable*** at ***γ* < 1** (**XPM/SPM < 1**), but they are ***unstable*** at ***γ* > 1** .

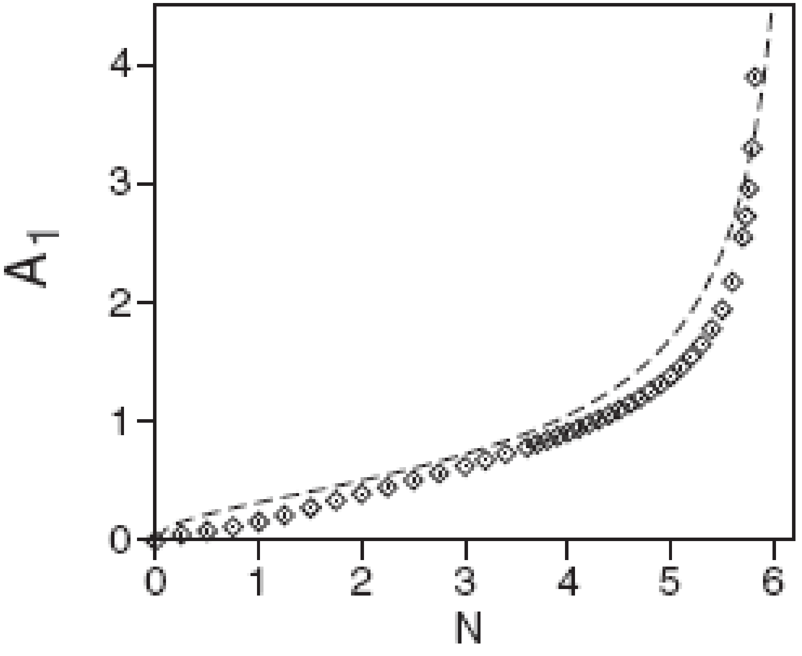
In the limit of $\mathbf{N \rightarrow N_{critical} \approx 5.85}$, the semi-vortex ***degenerates*** into the usual (unstable) ***Townes' soliton*** with an ***infinitely large chemical potential***, in the first component, leaving the second (formerly vortical) component ***empty***:



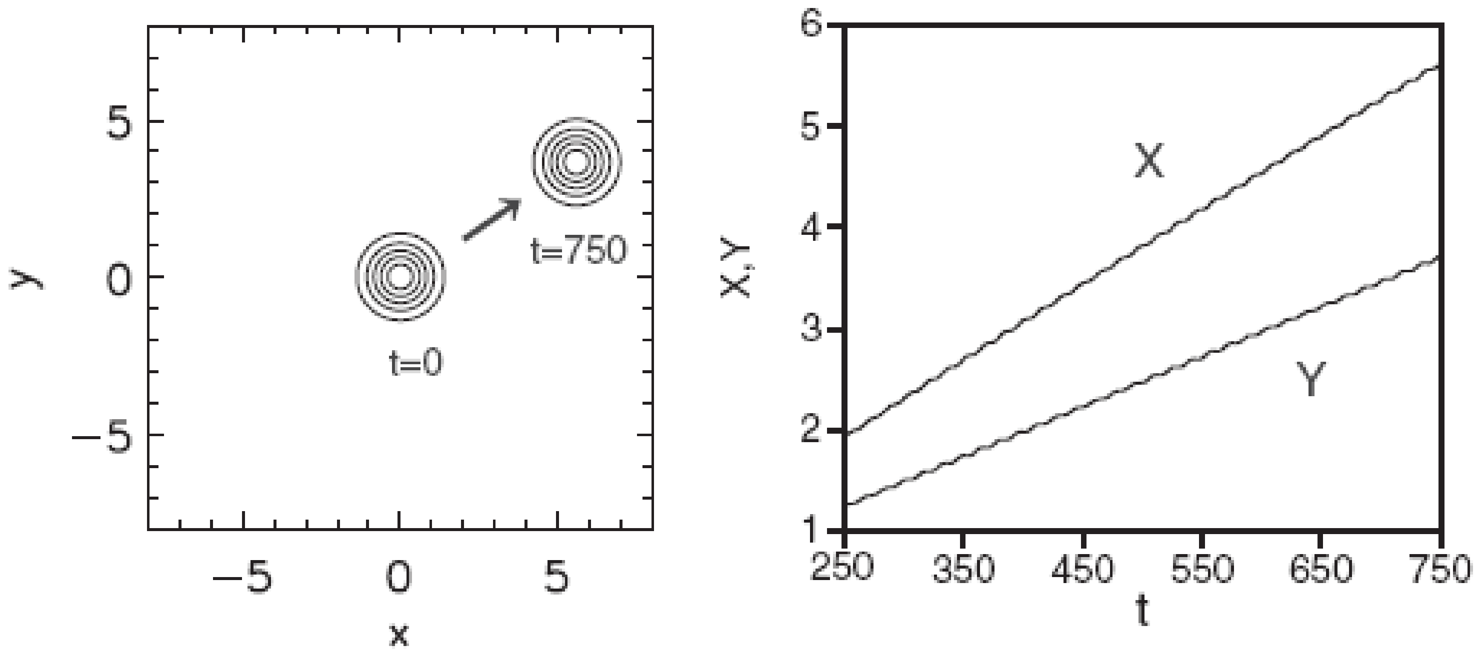
The variational approximation (**VA**) for semi-vortices was developed too, based on ansatz

$$\phi_+ = A_1 \exp\left(-i\mu t - \alpha_1 r^2\right), \; \phi_- = A_2 r \exp\left(-i\mu t + i\theta - \alpha_1 r^2\right),$$

which yields **good accuracy**, in comparison with numerical findings (dashed - the **VA** prediction, circles - numerical results):



Unstable semi-vortices at ***γ* > 1** commence ***spontaneous motion*** in the free space, keeping a robust shape in the process. An examples for ***γ* = 1** (the right panel shows the trajectory of the motion):



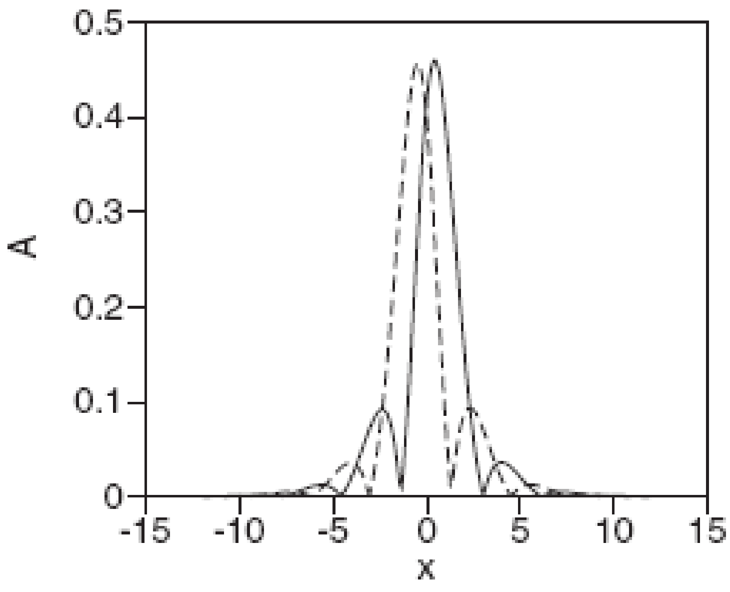
(4) Mixed modes

Another class of localized states can be constructed in the form of **mixed modes**, so called because they **mix** fundamental and vortical terms in each components, namely, $m_1 = (0,-1)$ and $m_2 = (0,+1)$, as per the following ansatz, which was used both as an input for the imaginary-time simulations and for the **VA**:

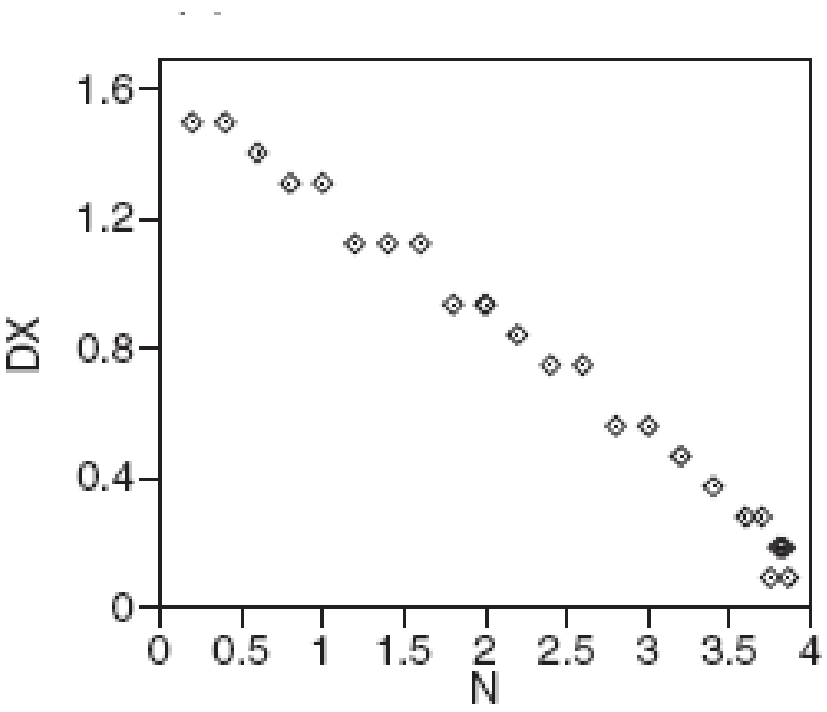
$$\phi_+ = A_1 \exp\left(-\alpha_1 r^2\right) - A_2 r \exp\left(-i\theta - \alpha_2 r^2\right),$$

$$\phi_- = A_1 \exp\left(-\alpha_1 r^2\right) + A_2 r \exp\left(+i\theta - \alpha_2 r^2\right).$$

A typical example of the cross-section of the mixed mode:

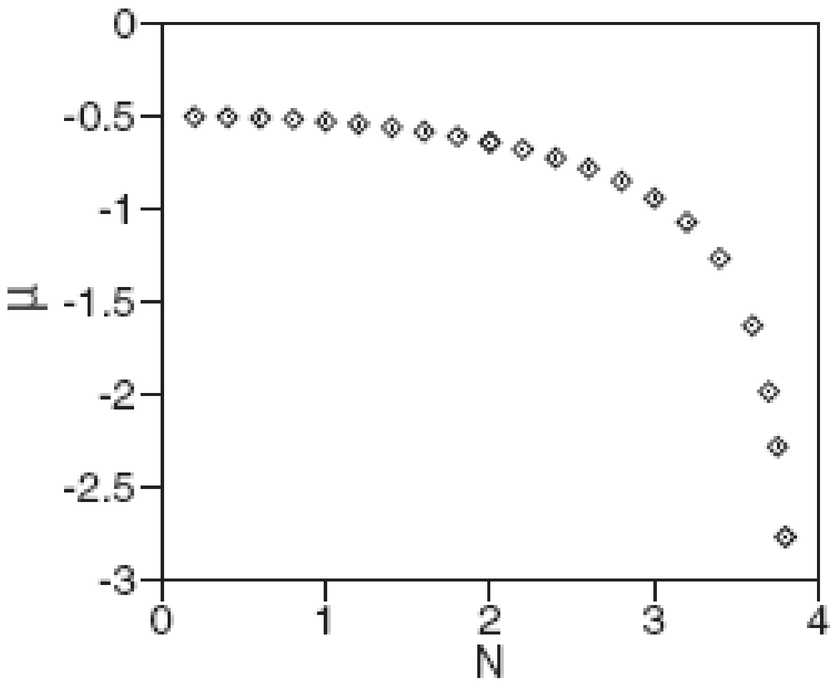


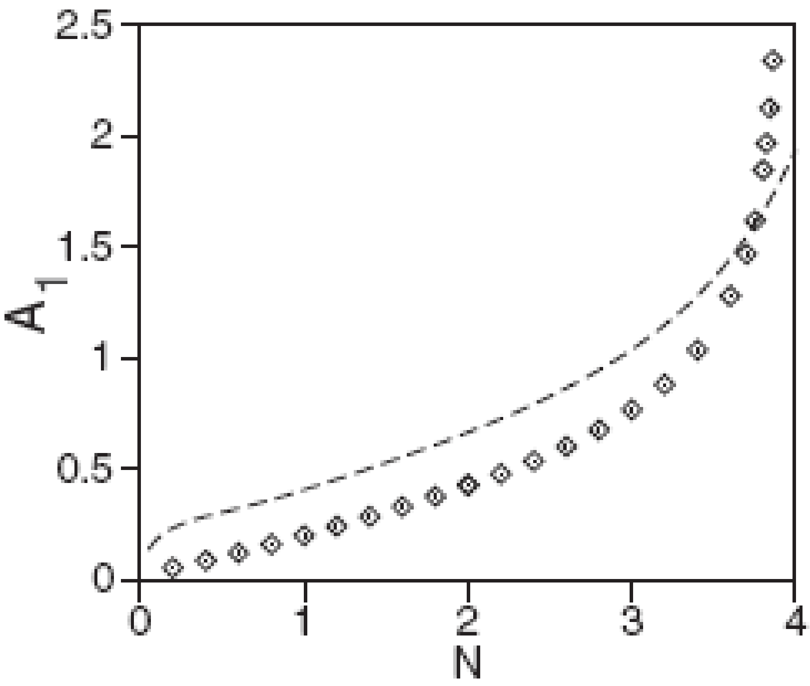
A peculiarity of the ***mixed mode*** is the **splitting**, **DX**, between maxima of the two components, in the *x* direction:



The dependence between the chemical potential and norm demonstrates that the **mixed-mode** family also complies with the **VK criterion**, hence it may be stable. In the limit of $\mathbf{N \rightarrow N_{critical} \approx 2 \cdot 5.85/(1 + \gamma)}$, the mixed mode ***degenerates*** into a two-component ***Townes' soliton***, while the vortical terms in both components ***vanish***. The **VA** provides

reasonable accuracy in this case too (the right panel):

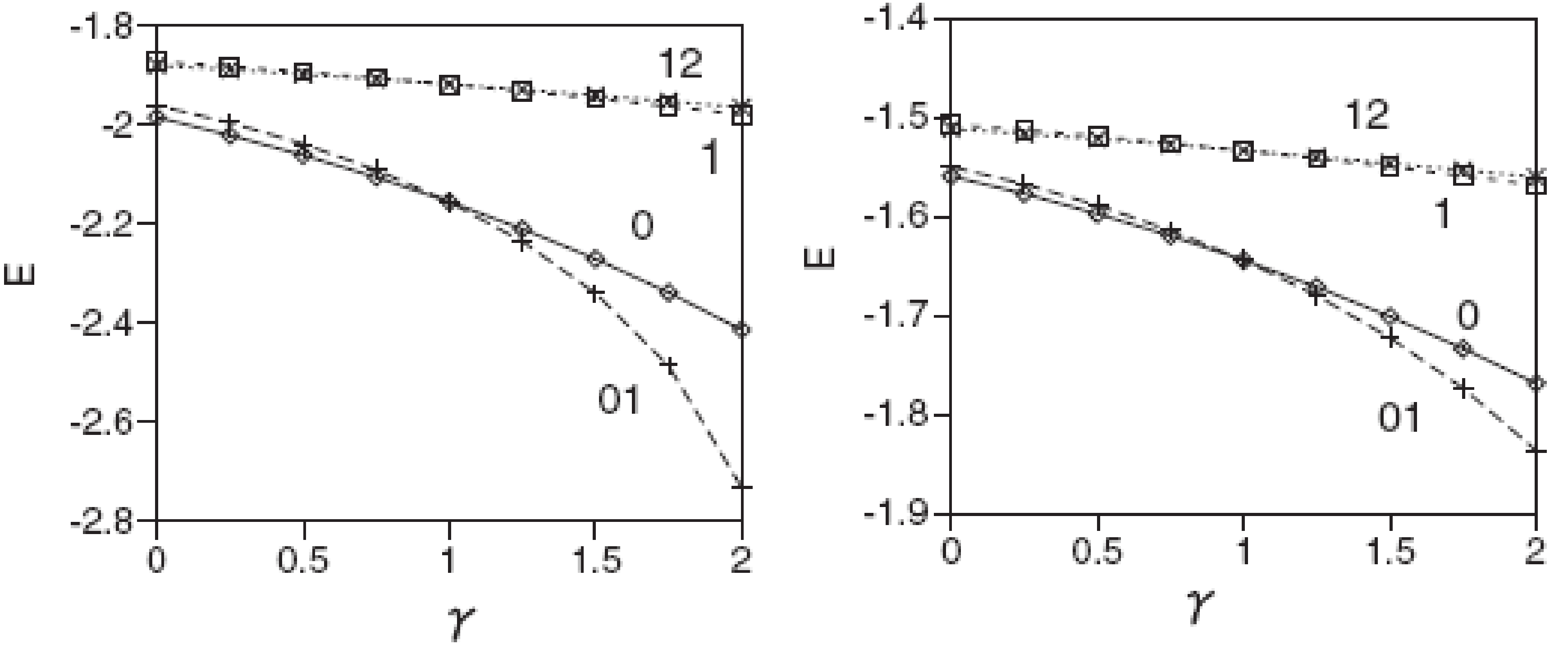




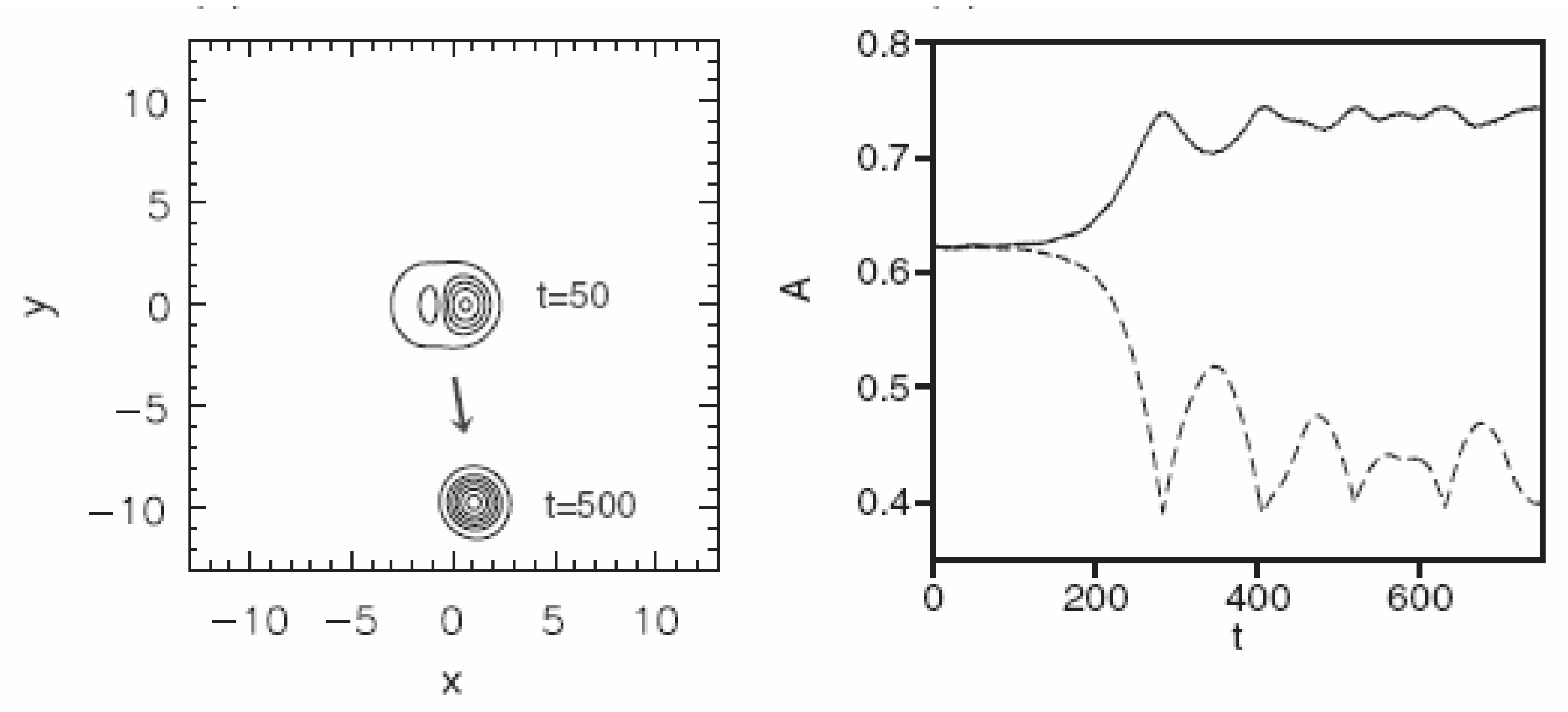
Direct simulations demonstrate that the mixed mode is ***unstable*** at $\gamma < 1$, and ***stable*** at $\gamma > 1$, i.e., exactly where the semi-vortex is ***stable*** or ***unstable***, respectively.
This ***stability switch*** between the **semi-vortex** and **mixed mode** with the increase of $\gamma \equiv$ **XPM/SPM** is explained by the fact that the **semi-vortex** and **mixed mode** are ***ground states***, which realize the ***minimum*** of the system's energy, E, precisely at $\gamma < 1$ and $\gamma > 1$, respectively.

$$
\begin{aligned}
E = \iint \Bigg\{ & \frac{1}{2}(|\nabla \phi_+|^2 + |\nabla \phi_-|^2) - \frac{1}{2}(|\phi_+|^4 + |\phi_-|^4) \\
& - \gamma |\phi_+|^2 |\phi_-|^2 + \frac{\lambda}{2}\left[\phi_+^* \left(\frac{\partial \phi_-}{\partial x} - i \frac{\partial \phi_-}{\partial y} \right) \right. \\
& \left. + \phi_-^* \left(-\frac{\partial \phi_+}{\partial x} - i \frac{\partial \phi_+}{\partial y} \right) \right] + \mathrm{c.c.} \Bigg\} dx dy,
\end{aligned}
$$

The dependence of the **energy** of the semi-vortex ("**0**") and mixed mode ("**01**") on ***γ* ≡ XPM/SPM,** for two different fixed values of the total norm, **N = 3.7**, and **N = 3.0**:



At γ **< 1**, **unstable** mixed modes commence ***spontaneous motion*** and gradually transform into ***semi-vortices***, making amplitudes of the two components ***unequal***, see an example for γ **= 0**:



(6) Excited states

Adding extra vorticity, ***S***, to both components of the ***semi-vortex*** makes it an ***excited state***. An ***exact ansatz*** for the excited states is available:

$$\phi_+(x,y,t)=e^{-i\mu t+iS\theta}f_1(r^2),\ \phi_-(x,y,t)=e^{-i\mu t+i(1+S)\theta}r^{1+S}f_2(r^2),$$

with real functions $f_{1,2}(r^2)$ satisfying the following equations:

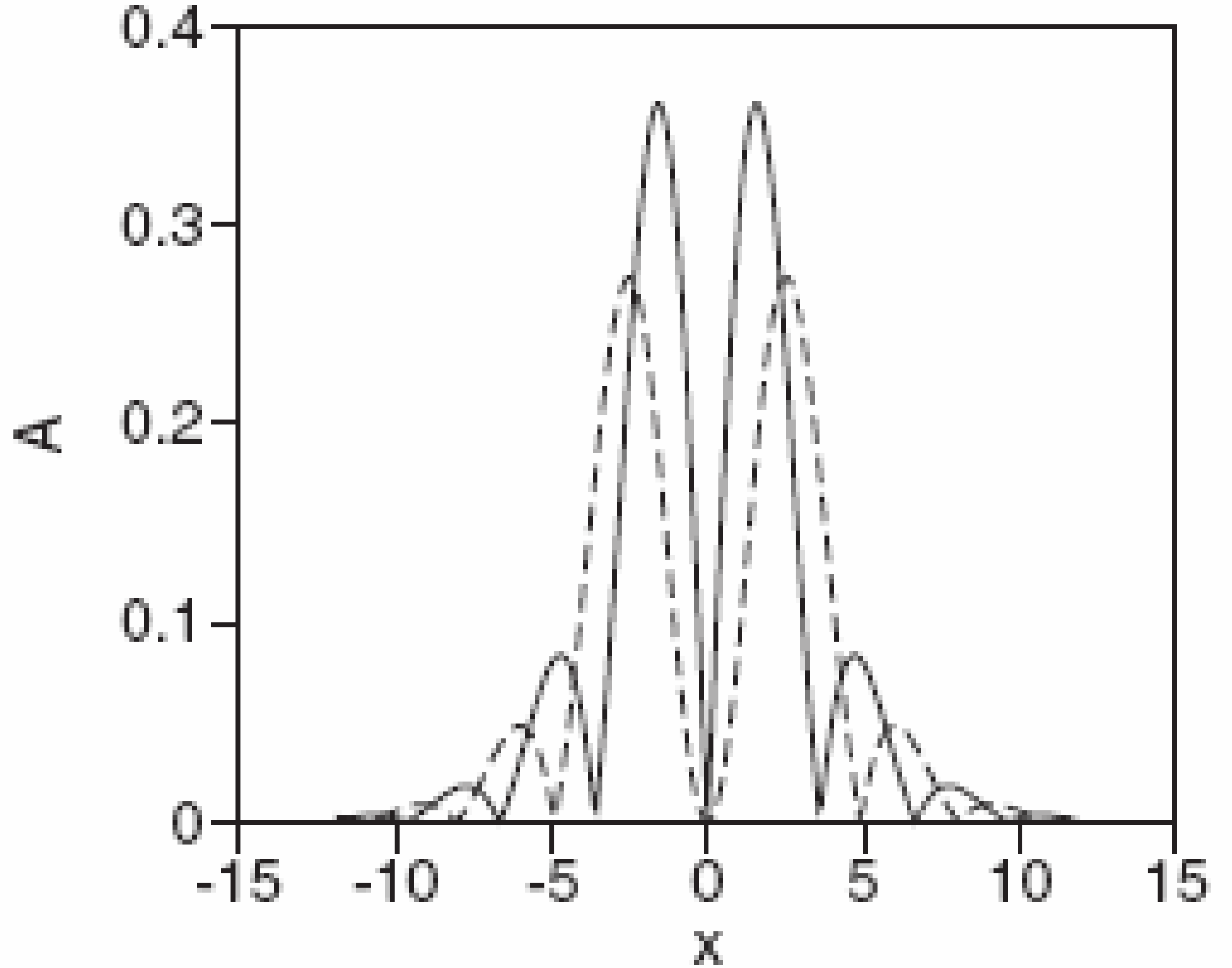
$$\mu f_1+2\left[r^2\frac{d^2f_1}{d(r^2)^2}+(1+S)\frac{df_1}{d(r^2)}\right]+r^{2S}\left(f_1^2+\gamma r^2f_2^2\right)f_1$$

$$-2\lambda\left[r^2\frac{df_2}{d(r^2)}+(1+S)f_2\right]-\frac{\Omega^2}{2}r^2f_1=0,$$

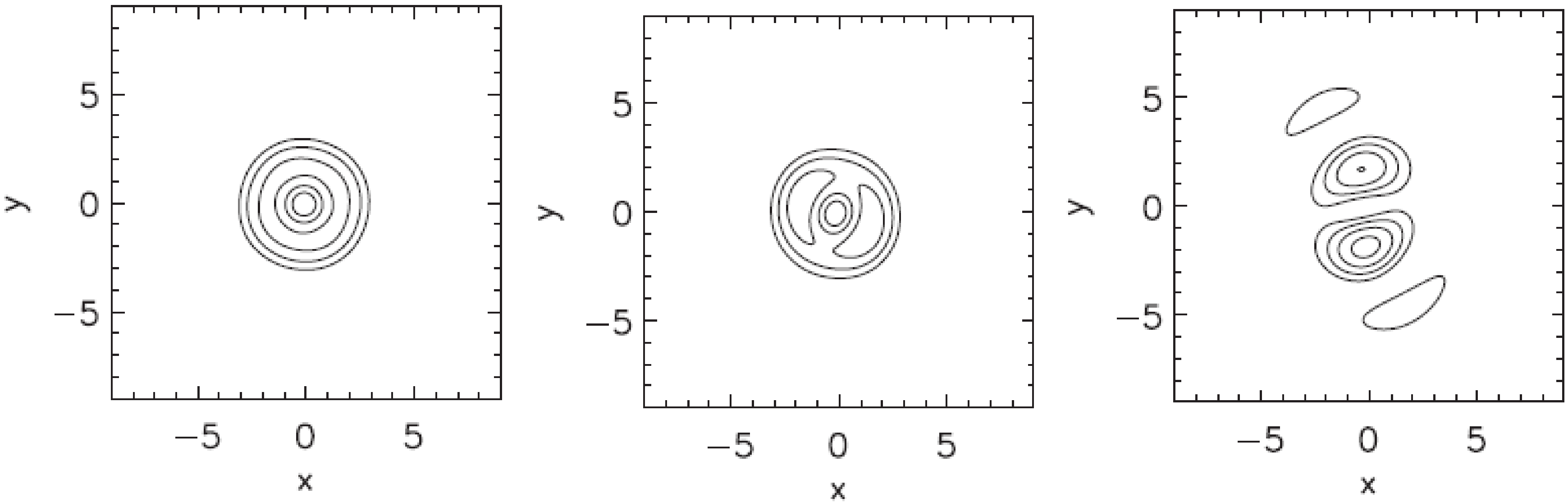
$$\mu f_2+2\left[r^2\frac{d^2f_2}{d(r^2)^2}+(2+S)\frac{df_1}{d(r^2)}\right]+r^{2S}\left(r^2f_2^2+\gamma f_1^2\right)f_2$$

$$+2\lambda\frac{df_1}{d(r^2)}-\frac{\Omega^2}{2}r^2f_2=0.$$

A example of the cross sections of components $|\Phi_+(x,0)|$ and $|\Phi_-(x,0)|$ (solid and dashed, respectively) in the excited state with $S = 1$, numerically found for $\gamma = 0$ and $N = 5$:



The ***excited states*** are ***completely unstable***. Typically, the evolution splits the former vortex, as show here for ***S* = 1**, ***N* = 3.7**, and ***γ* = 0**:



Excited states of the ***mixed mode*** can be found too, starting with the following input:

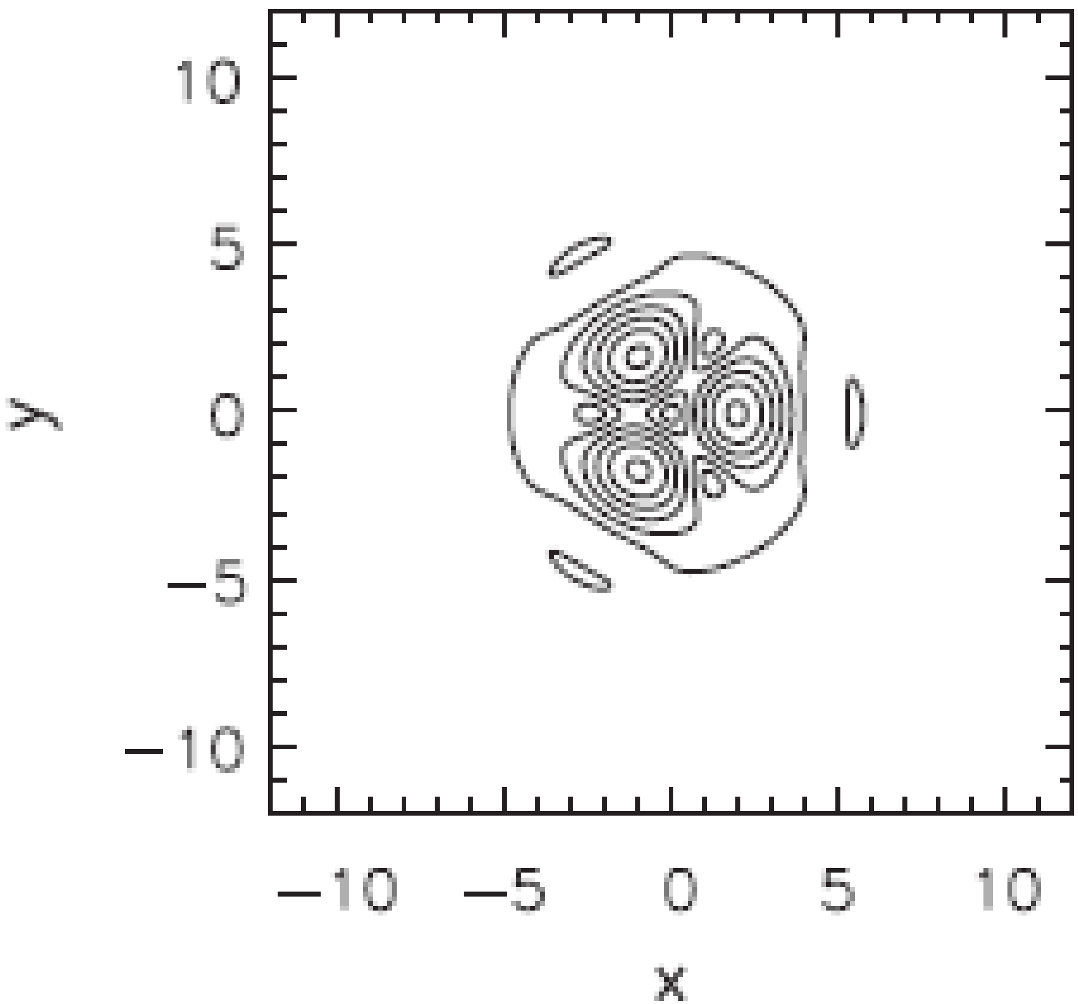
$$\phi_+ = A_1 r e^{i\theta} e^{-\alpha_1 r^2 - i\mu t} - A_2 r^2 e^{-2i\theta} e^{-\alpha_2 r^2 - i\mu t},$$
$$\phi_- = A_1 r e^{-i\theta} e^{-\alpha_1 r^2 - i\mu t} + A_2 r e^{2i\theta} e^{-\alpha_2 r^2 - i\mu t}. \quad (17)$$

This input can be also cast into the form

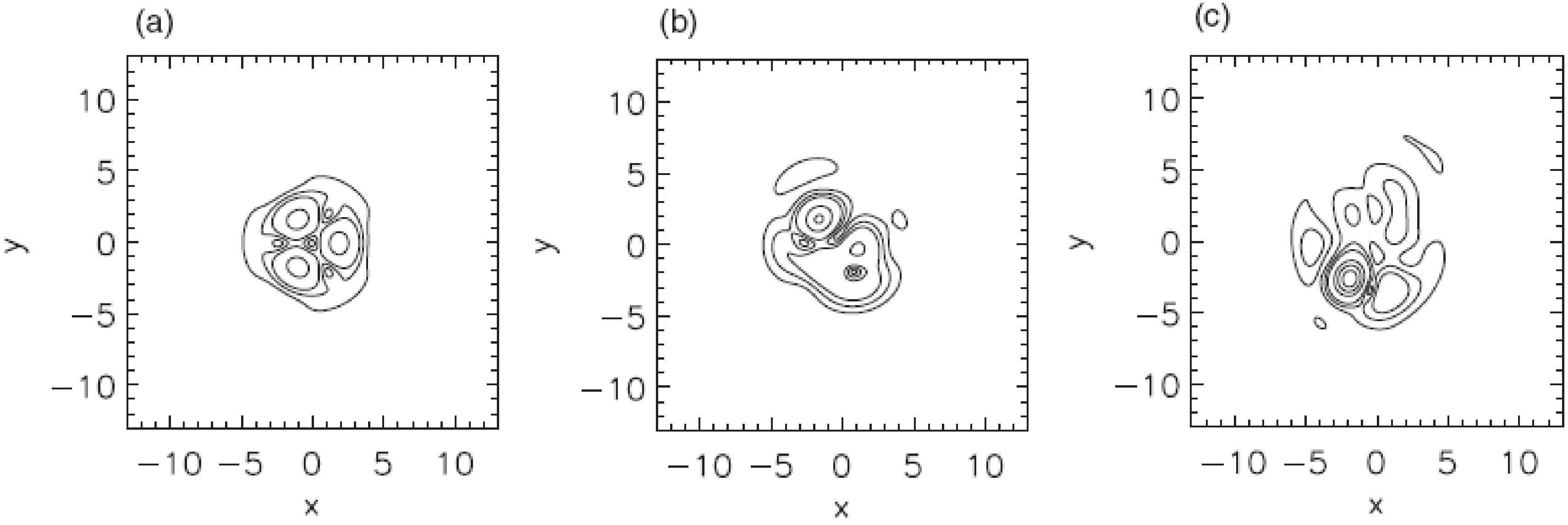
$$\phi_+ = r e^{i\theta} (A_1 e^{-\alpha_1 r^2 - i\mu t} - A_2 r^2 e^{-3i\theta} e^{-\alpha_2 r^2 - i\mu t}),$$
$$\phi_- = e^{-i\theta} (A_1 e^{-\alpha_1 r^2 - i\mu t} + A_2 r^2 e^{3i\theta} e^{-\alpha_2 r^2 - i\mu t}), \quad (18)$$

which implies that it includes a vortex with topological charge 1 set at $(x, y) = (0,0)$, and three vortices with charges -1 surrounding the origin. Figure 4(b) corroborates this

The top view of component **|$\boldsymbol{\Phi_+(x,y)}$|** of the excited state of the mixed mode for ***γ* = 2** and ***N* = 3**:



This species of the excited states is ***completely unstable*** too, being destroyed by the instability:



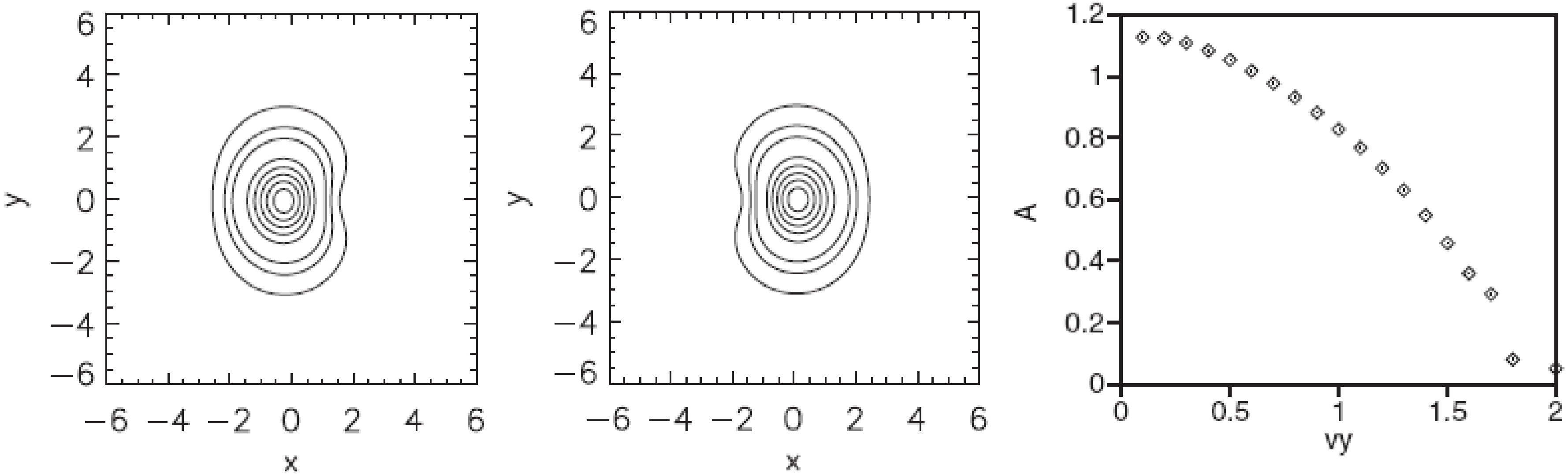
(6) Moving composite solitons and collisions between them

The ***mobility*** of the **2D** solitons in this system is a nontrivial issue, as the equations ***do not*** feature the Galilean invariance. Solutions for the ***mixed modes***, moving along axis ***y*** (but not along ***x***), with velocities $\boldsymbol{v_y < (v_y)_{max}}$, can be found as numerical solutions of the underlying GPEs, written in the ***moving reference frame***:

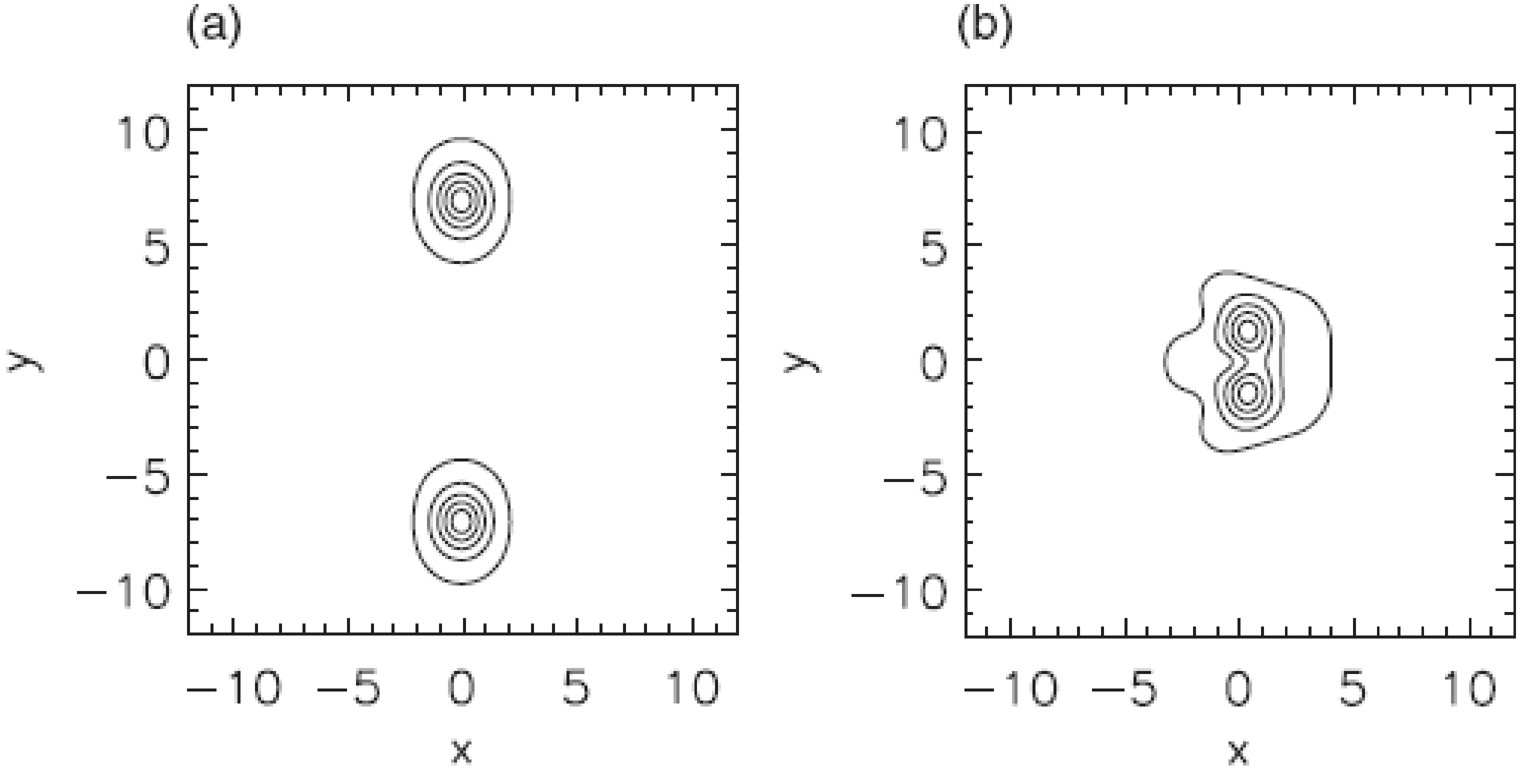
$$i\frac{\partial \phi_+}{\partial t} = -\frac{1}{2}\nabla^2\phi_+ - (|\phi_+|^2 + \gamma|\phi_-|^2)\phi'_+ + \lambda\left(\frac{\partial \phi_-}{\partial x} - i\frac{\partial \phi_-}{\partial y}\right) + \lambda v_y \phi_-,$$

$$i\frac{\partial \phi'_-}{\partial t} = -\frac{1}{2}\nabla^2\phi_- - (|\phi_-|^2 + \gamma|\phi_+|^2)\phi_- + \lambda\left(-\frac{\partial \phi_+}{\partial x} - i\frac{\partial \phi_+}{\partial y}\right) + \lambda v_y \phi_+.$$

For instance, for $\boldsymbol{N}$ **= 3.1**, $\boldsymbol{v}_{\mathbf{max}} \approx$ **1.8**. **2D** profiles of the two components of the moving mixed mode are shown here for $\boldsymbol{v}$ **= 0.5**. Semi-vortices may be mobile too, but with a ***very small*** limit velocity, $\mathbf{v}_{\mathbf{max}} \approx$ **0.03**.



Colliding moving mixed modes ***merge*** into a ***single*** solitary mode (shown here is the collision with velocities **±0.5**):



(6) Conclusions

The main result reported in this work is that the system of two **2D GPEs** with the self-attracting nonlinearity, coupled by the linear **SO** terms of the ***Rashba type***, gives rise to two families of ***composite*** (half-fundamental, half-vortical) solitons: **semi-vortices**, which are ***stable***, and realize the ***ground state*** of the system, for $\gamma \equiv$ **XPM/SPM < 1**, and **mixed modes**, which do the same for $\gamma > 1$.

To the best of our knowledge, this is the ***first example*** of a model in which solitons, supported by the ***cubic self-focusing*** in the **2D** geometry, are ***stable***. This may be explained by the fact that the solitons exist with values of the total norm ***below the critical value*** necessary for the onset of the ***collapse***. In the limit of the norm approaching the critical value, the solitons degenerate into unstable ***Townes' solitons***.

Both the semi-vortices and mixed modes may be predicted in an accurate form by means of the ***variational approximation***.

Excited states of the semi-vortices and mixed modes were found too, but they are ***completely unstable***.

In addition, the mixed-mode solitons may exist in a ***robust moving state***, up to a ***critical value*** of the velocity. ***Collisions*** between them lead to the merger into a ***single localized state***.

Finally, preliminary analysis demonstrates that this stabilization mechanism ***does not*** work in the **3D** setting, where the collapse is ***supercritical***, hence the corresponding critical norm is **N = 0**.