Fermionization, bosonization, and correlations in ultracold Bose and Fermi gases in tight waveguides

Marvin D. Girardeau
School of Optical Sciences, University of Arizona, Tucson, AZ

Recent collaborators: Anna Minguzzi
Adolfo del Campo and Gonzalo Muga
Ewan Wright, Gregory Astrakharchik

$$: Office of Naval Research/U. of Southern California, Army Research Office
Ultrascold gases in tight de Broglie waveguides

History (TG gas), experimental realization

Fermi-Bose mapping: Exact many-body solution for TG gas

Fermions with strong attractions: FTG gas

Exact solution, superconductive long-range order

Generalized FB mapping: Spin-aligned fermions ↔ bosons

Bose gases A and B with FTG AB interactions

Spinor fermions ↔ Lieb-Liniger-Heisenberg gas

Local two and three-body correlations, stability, photoassociation
First treatment of 1D hard-sphere gas: L. Tonks, Phys. Rev. 50, 955 (1936)

High temperature, inapplicable to ultracold atomic vapors
TG gas history

- First treatment of 1D hard-sphere gas: L. Tonks, Phys. Rev. 50, 955 (1936)
  
  *High temperature, inapplicable to ultracold atomic vapors*

- Exact ground state energy: A. Bijl, Physica 4, 329 (1937), Note II
  
  *No derivation*
First treatment of 1D hard-sphere gas: L. Tonks, Phys. Rev. 50, 955 (1936)

High temperature, inapplicable to ultracold atomic vapors

Exact ground state energy: A. Bijl, Physica 4, 329 (1937), Note II

No derivation

First published derivation:

TG gas history

- First treatment of 1D hard-sphere gas: L. Tonks, Phys. Rev. 50, 955 (1936)
  
  *High temperature, inapplicable to ultracold atomic vapors*

- Exact ground state energy: A. Bijl, Physica 4, 329 (1937), Note II
  
  *No derivation*

- First published derivation:
  

- Later independent derivations:
  
  H. Stachowiak, Acta Univ. Wratislaviensis 12, 93 (1960)
  
  M. Girardeau, J. Math. Phys. 1, 516 (1960) *Fermi-Bose mapping, both ground and excited states*
Low-dimensional systems

- Put ultracold vapor in long, narrow trap with transverse oscillation energy quantum > longitudinal energy
  ⇒ Transverse excitations frozen ⇒ 1D dynamics
  - Enhanced interactions and correlations
  - Failure of mean field theories
  - New quantum phases

- Experimental realization: optical lattices
Experimental realization of TG gas

TG gas ≡ 1D Bose gas with very strong repulsive interactions
⇒ dimensionless coupling constant $\gamma_{1D} \to \infty$
$\gamma_{1D} = \frac{E_{\text{int}}}{E_{\text{kin}}} = \frac{mg_{1D}}{\hbar^2 n} \Rightarrow$ TG limit reached at large
1D coupling constant $g_{1D}$ or low 1D density $n$ or large effective
mass $m$

Experiments:
How to increase $g_{1D}$

1D $s$-wave scattering length can be tuned by applying an external magnetic field.

Resonance due to approach of a bound state to the continuum.

- Tune to 1D confinement induced resonance: M. Olshanii, Phys. Rev. Lett. 81, 938 (1998)
Paredes et al. increased the effective mass $m$ by applying an optical lattice in the longitudinal direction. Then actual mass $m$ is replaced by solid state effective mass $m^* \gg m$. 
Paredes et al. experimental $n(p)$

- Black: TG theory, Dots: Experiment
- Green: Ideal Bose gas
- Yellow: Ideal Fermi gas
- Inset: Theoretical densities (black: invalid Thomas-Fermi)
Kinoshita et al. experiment

- No optical lattice, measure longitudinal cloud width vs. $\gamma_{1D}$
- Red: TG theory
- Barred dots: experiment
- Long dashes: mean field
Atoms in tight waveguide $\hbar \omega_{\perp} \gg \hbar \omega_{\ell}, k_BT, \mu$

Interactions: atom-atom scattering in transverse harmonic confinement
### TG gas theory: FB mapping

**Atoms in tight waveguide** $\hbar \omega_\perp \gg \hbar \omega_\perp, k_B T, \mu$

**Interactions**: atom-atom scattering in transverse harmonic confinement

**Bosons** [M. Olshanii, PRL 81, 938 (1998)]:

$$g_{1D} = 2a_s \hbar \omega_\perp (1 - 1.460 a_s/a_\perp)^{-1}$$

$$a_s = \lim_{k \to 0} \tan \delta_s(k)/k = 3D s\text{-wave scattering length, } a_\perp = \sqrt{2\hbar/m\omega_\perp}$$
TG gas theory: FB mapping

- Atoms in tight waveguide $\hbar \omega_\perp \gg \hbar \omega_\ell, k_B T, \mu$
  
  Interactions: atom-atom scattering in transverse harmonic confinement

- **Bosons** [M. Olshanii, PRL 81, 938 (1998)]:
  
  $$g_{1D} = 2a_s \hbar \omega_\perp (1 - 1.460 a_s / a_\perp )^{-1}$$
  
  $$a_s = \lim_{k \to 0} \tan \delta_s(k)/k = 3D \text{ s-wave scattering length},
  a_\perp = \sqrt{2 \hbar / m \omega_\perp}$$

- Feshbach resonance tune: $g_{1D} \to +\infty$ (TG gas) as $a_s / a_\perp \to .6848$
Atoms in tight waveguide $\hbar \omega_{\perp} \gg \hbar \omega_\ell, k_B T, \mu$

Interactions: atom-atom scattering in transverse harmonic confinement

**Bosons** [M. Olshanii, PRL 81, 938 (1998)]:

$$g_{1D} = 2 a_s \hbar \omega_{\perp} (1 - 1.460 \frac{a_s}{a_{\perp}})^{-1}$$

$$a_s = \lim_{k \to 0} \tan \delta_s (k)/k = 3D \text{ s-wave scattering length}, \quad a_{\perp} = \sqrt{2\hbar/m \omega_{\perp}}$$

**Feshbach resonance tune:** $g_{1D} \to +\infty$ (TG gas) as $a_s/a_{\perp} \to .6848$

**Folk theorem:** Fermi ground state energy > Bose ground state energy due to exclusion principle
Atoms in tight waveguide \( \hbar \omega_\perp \gg \hbar \omega_\parallel, k_B T, \mu \)

Interactions: atom-atom scattering in transverse harmonic confinement

**Bosons** [M. Olshanii, PRL 81, 938 (1998)]:

\[
g_{1D} = 2a_s \hbar \omega_\perp \left(1 - 1.46 a_s/a_\perp\right)^{-1}
\]

\[
a_s = \lim_{k \to 0} \tan \delta_s(k)/k = 3D \text{ s-wave scattering length}, a_\perp = \sqrt{2\hbar/m\omega_\perp}
\]

Feshbach resonance tune: \( g_{1D} \to +\infty \) (TG gas) as \( a_s/a_\perp \to 0.6848 \)

**Folk theorem:** Fermi ground state energy > Bose ground state energy due to exclusion principle

*FALSE* if motion is 1D AND interactions have hard core (TG)
True theorem: For 1D hard-core particles complete Bose and Fermi energy spectra are identical: “Fermi-Bose duality”
True theorem: For 1D hard-core particles complete Bose and Fermi energy spectra are identical: “Fermi-Bose duality”

True for arbitrary interactions $v_{int}(x_j - x_k)$ and arbitrary external potentials $v_{ext}(x_j)$ so long as $v_{int}$ has hard core forcing wave functions to vanish when cores overlap:

$$\psi(x_1, \ldots, x_N) = 0 \text{ if any } |x_j - x_k| < a$$
True theorem: For 1D hard-core particles complete Bose and Fermi energy spectra are identical: “Fermi-Bose duality”

True for arbitrary interactions \( v_{int}(x_j - x_k) \) and arbitrary external potentials \( v_{ext}(x_j) \) so long as \( v_{int} \) has hard core forcing wave functions to vanish when cores overlap:

\[
\psi(x_1, \cdots, x_N) = 0 \text{ if any } |x_j - x_k| < a
\]

Generalizes to time-dependent problems: Theorem: If \( v_{ext} \) is time-dependent and/or initial wave function not stationary (not an energy eigenstate), time-dependent single-particle densities \( n(x, t) \) generated by time-dependent Schrödinger equation for Bose and Fermi wave functions \( \psi_B \) and \( \psi_F \) are equal:

\[
n_B(x, t) = n_F(x, t)
\]

so long as motion is 1D, Bose and Fermi Hamiltonians are equal, and interaction contains a hard core.
How can these theorems be true? Exclusion principle *does* raise Fermi energy since only one particle allowed per orbital.
How can these theorems be true? Exclusion principle *does* raise Fermi energy since only one particle allowed per orbital.

Answer: *Hard core exclusion* raises the Bose energy, and one can *trade energy increase due to hard core exclusion for energy increase due to Pauli exclusion.*
How can these theorems be true? Exclusion principle *does* raise Fermi energy since only one particle allowed per orbital.

Answer: *Hard core exclusion* raises the Bose energy, and one can *trade energy increase due to hard core exclusion for energy increase due to Pauli exclusion*.

Proof of theorems: *FB mapping*

\[
\psi_B(x_1, \cdots, x_N; t) = A(x_1, \cdots, x_N)\psi_F(x_1, \cdots, x_N; t) \quad \text{where}
\]

\[
A(x_1, \cdots, x_N) = \prod_{1 \leq j < k \leq N} \text{sgn}(x_j - x_k) (\pm 1 \text{ everywhere, changes sign under particle exchange } x_j \leftrightarrow x_k.)
\]

Here

\[
\text{sgn}(x) = +1 (-1) \text{ if } x > 0 (x < 0).
\]
How can these theorems be true? Exclusion principle *does* raise Fermi energy since only one particle allowed per orbital.

Answer: *Hard core exclusion* raises the Bose energy, and one can *trade energy increase due to hard core exclusion for energy increase due to Pauli exclusion*.

Proof of theorems: *FB mapping*

\[ \psi_B(x_1, \cdots, x_N; t) = A(x_1, \cdots, x_N) \psi_F(x_1, \cdots, x_N; t) \]

where \( A(x_1, \cdots, x_N) = \prod_{1 \leq j < k \leq N} \text{sgn}(x_j - x_k) (\pm 1 \text{ everywhere, changes sign under particle exchange } x_j \leftrightarrow x_k.) \)

Here \( \text{sgn}(x) = +1 (-1) \) if \( x > 0 (x < 0) \).

Hamiltonian commutes with \( A \) since \( A \) constant except at points \( x_j = x_k \) where \( \psi = 0 \).
If there are interactions outside the hard cores, then FB mapping transforms a not exactly soluble many-boson problem into a not exactly soluble many-fermion problem, but it can be useful for approximate calculations.
If there are interactions outside the hard cores, then FB mapping transforms a not exactly soluble many-boson problem into a not exactly soluble many-fermion problem, but it can be useful for approximate calculations.

If only hard core interactions with or without an external longitudinal potential then strongly interacting many-boson problem maps into noninteracting fermions $\Rightarrow$ exact ground and excited states expressible in terms of one-particle orbitals $\varphi_i(x_j)$ appropriate to boundary conditions:

- No external potential: plane waves.
- Harmonic longitudinal trap potential: Hermite-Gaussians
- Optical lattice: Bloch or Wannier orbitals.
TG gas theory: FB mapping

Zero hard-core diameter, \( a \rightarrow 0 \) (good approx. for low densities of ultracold vapors): 
\[
\psi_F = \det_{i,j=1}^{N} \varphi_i(x_j)
\]
Bosonic ground state: 
\[
\psi_{B0} = A\psi_{F0} = |\psi_{F0}|
\]
TG gas theory: FB mapping

Zero hard-core diameter, \( a \to 0 \) (good approx. for low densities of ultracold vapors): 
\[
\psi_F = \det_{i,j=1}^{N} \varphi_i(x_j)
\]
Bosonic ground state: 
\[
\psi_{B0} = A \psi_{F0} = |\psi_{F0}|.
\]

\( a > 0 \): 
\[
\psi_F = \det_{i,j=1}^{N} \varphi_i(w_j) \quad \text{where}
\]
\[
w_1 = x_1, w_2 = x_2 - a, \cdots, w_N = x_N - (N - 1)a.
\]
TG gas theory: FB mapping

- Zero hard-core diameter, $a \to 0$ (good approx. for low densities of ultracold vapors): $\psi_F = \det_{i,j=1}^N \phi_i(x_j)$

Bosonic ground state: $\psi_{B0} = A \psi_{F0} = |\psi_{F0}|$.

- $a > 0$: $\psi_F = \det_{i,j=1}^N \phi_i(w_j)$ where
  
  $w_1 = x_1, w_2 = x_2 - a, \ldots, w_N = x_N - (N - 1)a$.

- “Fermionization” occurs for properties expressible in terms of configurational probability density:

  $|\psi_{B0}(x_1, \ldots, x_N)|^2 = |\psi_{F0}(x_1, \ldots, x_N)|^2$. 
TG gas theory: FB mapping

- Zero hard-core diameter, $a \to 0$ (good approx. for low densities of ultracold vapors): $\psi_F = \det_{i,j=1}^{N} \varphi_i(x_j)$

  Bosonic ground state: $\psi_{B0} = A \psi_{F0} = |\psi_{F0}|$.

- $a > 0$: $\psi_F = \det_{i,j=1}^{N} \varphi_i(w_j)$ where 
  
  $w_1 = x_1,$ $w_2 = x_2 - a, \cdots, w_N = x_N - (N - 1)a$.

- “Fermionization” occurs for properties expressible in terms of configurational probability density:
  
  $|\psi_{B0}(x_1, \cdots, x_N)|^2 = |\psi_{F0}(x_1, \cdots, x_N)|^2$.

- **Momentum distribution function** $n(k)$ can **not** be so expressed, so $n_B(k)$ is very different from $n_F(k)$.

  Lenard proved by *tour de force* that for TG gas
  
  $n_B(0) = \mathcal{O}(\sqrt{N})$, $\gg 1$ but $\ll \mathcal{O}(N)$ value required for BEC:

TG gas: Exact ground states

Simplest case: \( N \) bosons on ring of circumference \( L \), \( a \rightarrow 0 \), no external potential.


Van der Monde determinant of plane waves \( \Rightarrow \) Bijl-Jastrow form for ground state:

\[
\psi_{B0} = \prod_{i>j} | \sin[\pi L^{-1}(x_i - x_j)] |
\]

Low excited states have phonon character with sound speed
\( c = \frac{\pi \hbar n}{m} \) where \( n = \frac{N}{L} \).

\[ \psi_{B0}(x_1, \cdots, x_N) = C_N \left[ \prod_{i=1}^{N} e^{-Q_i^2/2} \right] \prod_{1 \leq j < k \leq N} |x_k - x_j| \]

with \( Q_i = x_i / x_{osc} \) and \( x_{osc} = \sqrt{\hbar/m\omega_{osc}} \).

\[
\psi_{B0}(x_1, \cdots, x_N) = C_N \left[ \prod_{i=1}^{N} e^{-Q_i^2/2} \right] \prod_{1 \leq j < k \leq N} |x_k - x_j|
\]

with \( Q_i = x_i / x_{osc} \) and \( x_{osc} = \sqrt{\hbar / m\omega_{osc}} \).

Momentum distribution very different from that of Fermi gas:

Bose and Fermi momentum distributions for \( N = 10 \). Solid line=Fermi, dashed line=Bose
The fermionic TG (FTG) gas

- FTG gas is *mirror image* of TG gas: Instead of *bosons* with infinitely strong *repulsive* interactions mapping to ideal *Fermi* gas, it has spin-aligned *fermions* with infinitely strong *attractive* interactions mapping to ideal *Bose* gas:

  M.D. Girardeau and M. Olshanii, cond-mat/0309396
FTG gas is mirror image of TG gas: Instead of bosons with infinitely strong repulsive interactions mapping to ideal Fermi gas, it has spin-aligned fermions with infinitely strong attractive interactions mapping to ideal Bose gas:
M.D. Girardeau and M. Olshanii, cond-mat/0309396

Interaction $v(x_j - x_k)$ is zero-range, infinite depth limit of square well of width $2x_0$ and depth $V_0$: $v(x) = -V_0$ if $-x_0 < x < x_0$ and $= 0$ if $|x| > x_0$
The fermionic TG (FTG) gas

- FTG gas is *mirror image* of TG gas: Instead of *bosons* with infinitely strong *repulsive* interactions mapping to ideal *Fermi* gas, it has spin-aligned *fermions* with infinitely strong *attractive* interactions mapping to ideal *Bose* gas:
  - M.D. Girardeau and M. Olshanii, cond-mat/0309396

- Interaction $v(x_j - x_k)$ is zero-range, infinite depth limit of square well of width $2x_0$ and depth $V_0$: $v(x) = -V_0$ if $-x_0 < x < x_0$ and $= 0$ if $|x| > x_0$

- Assume total energy zero $\Rightarrow$ exterior solution $\text{sgn}(x) = \pm 1$ and interior solution $\sin(\kappa x)$ with $\kappa = \sqrt{mV_0/\hbar^2} = \pi/2x_0$.
  - $x_0 \to 0$ and $V_0 \to \infty \Rightarrow V_0 x_0^2 \to (\pi \hbar)^2/8m = \text{const.} \Rightarrow$
  - *Stronger than attractive delta function*
FTG gas: Exact ground state

- Exact ground state for $N=2$, no trap: Zero-energy scattering solution for finite well width compared with zero-range limit (notation: $z = x$)

Green: FTG gas (zero range limit), Red: Nonzero range
FTG gas: Exact ground state

- Exact ground state for $N=2$, no trap: Zero-energy scattering solution for finite well width compared with zero-range limit (notation: $z = x$)

- FTG maps to *ideal Bose gas*. FTG ground state for arbitrary $N$: $\psi_F(x_1, \cdots, x_N) = A(x_1, \cdots, x_N) \prod_{j=1}^{N} \varphi_0(x_j)$ where $\varphi_0 = \text{constant for no trap, Gaussian for harmonic trapping}$
FTG gas: $\rho_1$ and $n_k$

- 1-particle density matrix $\rho_1$ and momentum distribution $n_k$ can be determined exactly:


In thermodynamic limit $N \to \infty$, $L \to \infty$, density $N/L \to n$ for untrapped gas of length $L$, this gives an exponential decay:

$$\rho_1(x, x') = \langle \Psi_0 | \hat{\psi}^\dagger(x) \hat{\psi}(x') | \Psi_0 \rangle = ne^{-2n|x-x'|}.$$
FTG gas: $\rho_1$ and $n_k$

- 1-particle density matrix $\rho_1$ and momentum distribution $n_k$ can be determined exactly:


  In thermodynamic limit $N \to \infty$, $L \to \infty$, density $N/L \to n$ for untrapped gas of length $L$, this gives an exponential decay:

  \[ \rho_1(x, x') = \langle \Psi_0 | \hat{\psi}^\dagger(x) \hat{\psi}(x') | \Psi_0 \rangle = ne^{-2n|x-x'|}. \]

- Fourier transform of $\rho_1$ is momentum distribution function

  \[ n_k = [1 + (k/2n)^2]^{-1} \]

  Satisfies exclusion principle $n_k \leq 1$, but for $n \to 0$ the continuous momentum density $n(k) = (L/2\pi)n_k \to N\delta(k) = \text{ideal Bose gas distribution (bosonization)}$
FTG gas: $\rho_2$ and superconductivity

- 2-particle density matrix $\rho_2$ can also be determined exactly:

Thermodynamic limit for untrapped FTG gas:

$$\rho_2(x_1, x_2; x'_1, x'_2) = n^2 \text{sgn}(x_1 - x_2)\text{sgn}(x'_1 - x'_2)e^{2n(y_1 - y_2 + y_3 - y_4)}$$

where $y_1 \leq y_2 \leq y_3 \leq y_4$ are the arguments $(x_1, x_2; x'_1, x'_2)$ in ascending order.
FTG gas: $\rho_2$ and superconductivity

- 2-particle density matrix $\rho_2$ can also be determined exactly:

Thermodynamic limit for untrapped FTG gas:

$$\rho_2(x_1, x_2; x'_1, x'_2) = n^2 \text{sgn}(x_1 - x_2) \text{sgn}(x'_1 - x'_2) e^{2n(y_1 - y_2 + y_3 - y_4)}$$

where $y_1 \leq y_2 \leq y_3 \leq y_4$ are the arguments $(x_1, x_2; x'_1, x'_2)$ in ascending order.

- It remains of order $n^2$ for arbitrarily large separation of the centers of mass $X = (x_1 + x_2)/2$ and $X' = (x'_1 + x'_2)/2$ of the pairs $(x_1, x_2)$ and $(x_1, x_2)$, Yang’s criterion for superconductive off-diagonal long-range order $\Rightarrow$ large eigenvalue $N/2$ of $\rho_2$ with eigenfunction $= \text{const.} \text{sgn}(x_1 - x_2) e^{-2n|x_1 - x_2|}$:

  C.N. Yang, Rev. Mod. Phys. 34, 694 (1962), Sec. 18 and Appendix A
Generalized FB mapping

Mapping \( \psi_B = A(x_1, \ldots, x_N)\psi_F \) and inverse (with \( A^{-1} = A \)) not restricted to TG or FTG gas; also maps bosons with repulsive interactions of any coupling constant \( g_{1D}^B = 2\hbar^2/m|a_{1D}| > 0 \) and spin-aligned fermions with reciprocal attractive coupling constant \( g_{1D}^F = 2\hbar^2|a_{1D}|/m \) such that 1D scattering lengths are equal: \( a_{1D}^B = a_{1D}^F = a_{1D} < 0 \):

M.D. Girardeau and M. Olshanii, cond-mat/0309396

Generalized FB mapping

Mapping $\psi_B = A(x_1, \cdots, x_N)\psi_F$ and inverse (with $A^{-1} = A$) not restricted to TG or FTG gas; also maps bosons with repulsive interactions of any coupling constant $g_{1D}^B = 2\hbar^2/m|a_{1D}| > 0$ and spin-aligned fermions with reciprocal attractive coupling constant $g_{1D}^F = 2\hbar^2|a_{1D}|/m$ such that 1D scattering lengths are equal: $a_{1D}^B = a_{1D}^F = a_{1D} < 0$:

M.D. Girardeau and M. Olshanii, cond-mat/0309396


Bose gas with interactions $g_{1D}^B \delta(x_j - x_k)$ solved exactly for untrapped gas by Bethe ansatz: E.H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963) but only gives energies.
**Generalized FB mapping**

- Mapping $\psi_B = A(x_1, \cdots, x_N)\psi_F$ and inverse (with $A^{-1} = A$) not restricted to TG or FTG gas; also maps bosons with repulsive interactions of any coupling constant $g_{1D}^B = 2\hbar^2/m|a_{1D}| > 0$ and spin-aligned fermions with reciprocal attractive coupling constant $g_{1D}^F = 2\hbar^2|a_{1D}|/m$ such that 1D scattering lengths are equal: $a_{1D}^B = a_{1D}^F = a_{1D} < 0$:

  - M.D. Girardeau and M. Olshanii, cond-mat/0309396


- Bose gas with interactions $g_{1D}^B \delta(x_j - x_k)$ solved exactly for untrapped gas by Bethe ansatz: E.H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963) but only gives energies.

- Mapping allows strongly interacting 1D Fermi gas to be treated via weakly interacting Bose gas and vice versa.
Fermionic wave function $\Psi_F$ and mapped bosonic wave function $\Psi_B$ as a function of $x_{12}$. The potential is a deep and narrow square well plus a point hard core, but in the zero-range limit its effect on $\Psi_B$ outside the well is the same as that of $V_B = g_B^{1D} \delta(x_{12})$. 
A and B ideal Bose gases


Bose-Bose duality mapping removes AB interactions as for pure FTG gas:


No trap, periodic boundary conditions, box length $L$:

Exact ground state $\Psi_0 = \Psi_{M0} M = M$ where $\Psi_{M0} = 1$
and $M$ is the mapping function

$$M(x_1, \cdots, x_{N_A}; y_1, \cdots, y_{N_B}) = \prod_{i=1}^{N_A} \prod_{j=1}^{N_B} \text{sgn}(x_i - y_j)$$

where $\text{sgn}(x)$ is $+1$ $(-1)$ if $x > 0$ ($x < 0$).

$M^2 = 1 \Rightarrow$ densities $n_A(x)$ and $n_B(y)$ are constant:

$n_A = N_A/L, n_B = N_B/L$. 
Off-diagonal density matrix elements nontrivial due to discontinuities in $M$:

Single-particle density matrices in thermodynamic limit:

\[ \rho_{1A}(x, x') = n_A e^{-2n_B |x-x'|}, \quad \rho_{1B}(y, y') = n_B e^{-2n_A |x-x'|}. \]

Fourier transforms are Lorentzian momentum distributions:

\[ n_{kA} = \frac{4n_A n_B}{4n_B^2 + k^2}, \quad n_{kB} = \frac{4n_A n_B}{4n_A^2 + k^2}. \]

Infinite AB attraction destroys BEC of A and B.
Two-particle density matrices, pairing, ODLRO:
For $x_1 < x_2 < x'_1 < x'_2$

$$\rho_{2AA}(x_1, x_2; x'_1, x'_2) = n^2_A e^{-2n_B |x_1 - x_2|} e^{-2n_B |x'_1 - x'_2|}.$$  

and for $y_1 < y_2 < y'_1 < y'_2$

$$\rho_{2BB}(y_1, y_2; y'_1, y'_2) = n^2_B e^{-2n_A |y_1 - y_2|} e^{-2n_A |y'_1 - y'_2|}.$$  

If pairs $(x_1, x_2)$ and $(x'_1, x'_2)$ are separated while keeping $|x_1 - x_2|$ and $|x'_1 - x'_2|$ fixed, then $\rho_{2AA}$ remains constant $\Rightarrow AA$-pair ODLRO; similarly there is $BB$-pair ODLRO.
For AB pair density matrix one finds

\[
\rho_{2AB}(x, y; x', y') = n_A n_B \text{sgn}(x - y)\text{sgn}(x' - y') \\
\times e^{-2n_B |x - x'|} e^{-2n_A |y - y'|}
\]

Exponential decrease as pairs \((x, y)\) and \((x', y')\) separate
\(\Rightarrow\) no AB pair ODLRO, although FTG AB-pair attraction induces AA and BB ODLRO.

Quantum phase transition: If AB even-wave repulsion as well as AB odd-wave attraction \(\Rightarrow\) quantum phase transition between a phase with AB contact nodes and one with no AB nodes. For details see the paper.
A and B Calogero-Sutherland (CS) Bose gases

M.D. Girardeau and GE. Astrakharchik, arXiv FILLIN and PRA (submitted)

CS interaction has repulsive inverse square potential:

\[ V_{CS}(x_i - x_j) = \frac{\hbar^2 \lambda(\lambda - 1)}{m(x_i - x_j)^2} \]

on the infinite line with or without harmonic trapping.

For trapping on a ring of circumference \( L \), this is replaced by

\[ V_{CS}(x_i - x_j) = \frac{\pi^2 \hbar^2}{mL^2 \sin^2[\pi(x_i - x_j)/L]} \cdot \frac{\lambda(\lambda - 1)}{mL^2} . \]

Bose-Bose duality mapping removes FTG AB interactions as for pure FTG gas and ideal Bose gas mixture.
Exact ground state of single-component CS gas known for both harmonic trap and ring geometries. For harmonic trap

\[ \Psi_0(x_1, \ldots, x_N) = \prod_{i=1}^{N} e^{-\frac{x_i^2}{2x_{osc}^2}} \prod_{1 \leq i < j \leq N} |x_i - x_j|^\lambda \]

where \( x_{osc} = \sqrt{\frac{\hbar}{m\omega}} \) is oscillator length.

For ring trap

\[ \psi^{CS}(x_1, \ldots, x_N) = \prod_{i<j} \left| \sin \left( \frac{\pi(x_i - x_j)}{L} \right) \right|^\lambda \]
CS model reduces to previously known results in several limits, thus interpolating between those limits: ideal Bose gas for $\lambda = 0$, TG gas for $\lambda = 1$, and classical crystal for $\lambda \to \infty$.

Exact ground state of AB mixture is

$$\Psi_0 A \Psi_0 B M(x_1, \cdots, x_{N_A}; y_1, \cdots, y_{N_B})$$

where $\Psi_0 A$ and $\Psi_0 B$ are ground states of pure CS gases with coupling constants $\lambda_A$ and $\lambda_B$, and $M$ is same Bose-Bose duality mapping function as before.

Integrands of all one- and two-particle density matrices known in exact analytical form, but the integrals are multidimensional and we have evaluated them by a Monte Carlo method.
An optically trapped 1D gas of spin-$\frac{1}{2}$ fermionic atoms has interactions arising from both space-even, spin-odd (s-wave, singlet) scattering and space-odd, spin-even (p-wave, triplet) scattering. p-wave interaction usually negligible because space-odd wave function vanishes at contact, but it can be enormously enhanced by a p-wave Feshbach resonance ⇒ both spatially even- and odd-wave 1D interactions.
An optically trapped 1D gas of spin-$\frac{1}{2}$ fermionic atoms has interactions arising from both space-even, spin-odd (s-wave, singlet) scattering and space-odd, spin-even (p-wave, triplet) scattering. p-wave interaction usually negligible because space-odd wave function vanishes at contact, but it can be enormously enhanced by a p-wave Feshbach resonance $\Rightarrow$ both spatially even- and odd-wave 1D interactions.

Even-wave interaction of usual Lieb-Liniger (LL) delta function form enhanced by spin-singlet projector $\hat{P}_{j\ell}^s$:

$$g_{1D}^e \delta(x_{j\ell}) \hat{P}_{j\ell}^s.$$
An optically trapped 1D gas of spin-$\frac{1}{2}$ fermionic atoms has interactions arising from both space-even, spin-odd (s-wave, singlet) scattering and space-odd, spin-even (p-wave, triplet) scattering. p-wave interaction usually negligible because space-odd wave function vanishes at contact, but it can be enormously enhanced by a p-wave Feshbach resonance ⇒ both spatially even- and odd-wave 1D interactions.

Even-wave interaction of usual Lieb-Liniger (LL) delta function form enhanced by spin-singlet projector $\hat{P}_s^{j\ell}$:

$g_{1D}^{e} \delta(x_{j\ell}) \hat{P}_s^{j\ell}.$

Odd-wave interaction has attractive deep and narrow well like FTG interaction, enhanced by spin-triplet projector $\hat{P}_t^{j\ell}$:

$v_{1D}^{o} (x_{j\ell}) \hat{P}_t^{j\ell}.$
Map by introducing a factor $\epsilon(x_{j\ell})$ only for odd-wave interaction $\Rightarrow$ all interactions of mapped Hamiltonian are LL delta functions enhanced by spin singlet and triplet projectors $\Rightarrow$ “Lieb-Liniger-Heisenberg” model ($\hbar = 2m = 1$)

$$\hat{H}_{LLH} = -\sum_{j=1}^{N} \partial_{x_j}^2 + \sum_{1 \leq j < \ell \leq N} \left[ \frac{3c + c'}{4} + (c - c') \hat{S}_j \cdot \hat{S}_\ell \right] \delta(x_{j\ell})$$
Map by introducing a factor $\epsilon(x_{j\ell})$ only for odd-wave interaction $\Rightarrow$ all interactions of mapped Hamiltonian are LL delta functions enhanced by spin singlet and triplet projectors $\Rightarrow$ “Lieb-Liniger-Heisenberg” model ($\hbar = 2m = 1$)

$$\hat{H}_{LLH} = - \sum_{j=1}^{N} \frac{\partial^2 x_j}{\partial x_j} + \sum_{1 \leq j < \ell \leq N} \left[ \frac{3c + c'}{4} + (c - c') \hat{S}_j \cdot \hat{S}_\ell \right] \delta(x_{j\ell})$$

$c = 2/|a_{1D}^o|$ and $c' = 2/|a_{1D}^e|$ where $a_{1D}^e$ and $a_{1D}^o$ are even- and odd-wave scattering lengths, and spin-spin interactions arise from spin singlet and triplet projectors which induce implicit spin-spin interactions.
Map by introducing a factor $\epsilon(x_{j\ell})$ only for odd-wave interaction $\Rightarrow$ all interactions of mapped Hamiltonian are LL delta functions enhanced by spin singlet and triplet projectors $\Rightarrow$ “Lieb-Liniger-Heisenberg” model ($\hbar = 2m = 1$)

$$\hat{H}_{LLH} = -\sum_{j=1}^{N} \partial_{x_j}^2 + \sum_{1 \leq j < \ell \leq N} \left[ \frac{3c+c'}{4} + (c - c') \hat{S}_j \cdot \hat{S}_\ell \right] \delta(x_{j\ell})$$

- $c = 2/|a_{1D}^o|$ and $c' = 2/|a_{1D}^e|$ where $a_{1D}^e$ and $a_{1D}^o$ are even- and odd-wave scattering lengths, and spin-spin interactions arise from spin singlet and triplet projectors which induce implicit spin-spin interactions.

- **Ground state phase diagram:** $c < c'$: ferromagnetic, $S = N/2$; $c > c'$: antiferromagnetic, $S = 0$; $c = c'$: highly degenerate, $S$ from 0 to $\frac{N}{2}$. 
Spinor fermions ↔ LLH model

- **Ferromagnetic phase**: $c < c'$, spin-aligned, $S = N/2$ ⇒ No quantum fluctuations of $\hat{S}_j \cdot \hat{S}_\ell = \frac{1}{4}$ ⇒ LLH Hamiltonian = LL Hamiltonian, ground state independent of even-wave interaction $c'$, exact solution given by LL Bethe ansatz.
Spinor fermions ↔ LLH model

- **Ferromagnetic phase:** $c < c'$, spin-aligned, $S = N/2 \Rightarrow$ No quantum fluctuations of $\hat{S}_j \cdot \hat{S}_\ell = \frac{1}{4} \Rightarrow$ LLH Hamiltonian = LL Hamiltonian, ground state independent of even-wave interaction $c'$, exact solution given by LL Bethe ansatz.

- **Antiferromagnetic phase:** $c > c'$, $S = 0$. Use variational ground state which is product of purely spatial state $\psi_{\text{space}}(x_1, \cdots, x_N)$ by purely spin state $\psi_{\text{spin}}(\sigma_1, \cdots, \sigma_N)$. Still incorporates some space-spin correlation via spin singlet and triplet projectors.

  Take $\psi_{\text{spin}}$ to be ground state of isotropic, antiferromagnetic Heisenberg Hamiltonian $\hat{H}_{\text{Heis}} = \sum_{j=1}^{N-1} \hat{S}_j \cdot \hat{S}_{j+1} + \hat{S}_N \cdot \hat{S}_1$, for which $E_{0,\text{Heis}}/N = \frac{1}{4} - \ln 2 \Rightarrow$ LL model again, but with $c_{LL} = c(1 - \ln 2) + c'\ln 2 \Rightarrow E_0(c, c') < E_{0,LL}(c_{LL})$. 
Spinor fermions ↔ LLH model

Vertical axis = $E_0/n^2N$, horizontal axis = $c/n$ for fixed $c'$ where $c = 2/|a_{1D}^0|$ and $c' = 2/|a_{1D}^e|$. Solid line: Exact ferromagnetic ($c < c'$) $E_0$, Dashed lines: Antiferromagnetic ($c > c'$) $E_0$ for $c'/n = 0, 2, 5, \text{ and } 10$ (bottom to top).
Local two-body correlation function \( g_2 = \langle \hat{\psi}^\dagger(x)^2 \hat{\psi}(x)^2 \rangle_0 \) of LL gas determines rate of photoassociation to excited diatomic molecules.

Ferromagnetic phase: Expectation value taken in LL ground state, which depends on \( \gamma = c/n \).

Antiferromagnetic phase: Use LL ground state again, but with \( \gamma \) replaced by \( \gamma(1 - \ln 2) + \gamma' \ln 2 \) with \( \gamma = c/n \), \( \gamma' = c'/n \).

\( g_2 \) of LL gas known exactly from Hellmann-Feynman theorem:
\[
g_2(\gamma)/n^2 = de(\gamma)/d\gamma \text{ where } e(\gamma) = n^{-2}(E_0/N)
\]

Local two-body correlation function \( g_2 = \langle \hat{\psi}^\dagger(x)^2 \hat{\psi}(x)^2 \rangle_0 \) of LL gas determines rate of photoassociation to excited diatomic molecules.

Ferromagnetic phase: Expectation value taken in LL ground state, which depends on \( \gamma = c/n \).

Antiferromagnetic phase: Use LL ground state again, but with \( \gamma \) replaced by \( \gamma(1 - \ln 2) + \gamma' \ln 2 \) with \( \gamma = c/n \), \( \gamma' = c'/n \).

\( g_2 \) of LL gas known exactly from Hellmann-Feynman theorem:
\[
g_2(\gamma)/n^2 = de(\gamma)/d\gamma \text{ where } e(\gamma) = n^{-2}(E_0/N)
\]


\( g_3 = \langle \hat{\psi}^\dagger(x)^3 \hat{\psi}(x)^3 \rangle_0 \) determines rate of three-body recombination. Also known exactly: V.V. Cheianov, H. Smith, and M.B. Zvoranov, Phys. Rev. A 73, 051604 (2006).
Spinor fermions, \( g_2 \) and \( g_3 \)

\[
g_2(\gamma, \gamma')/n^2 \quad \text{and} \quad g_3(\gamma, \gamma')/n^3
\]

FTG gas lies along the \( \gamma' \) axis and the quantum phase transition along the dashed line. Colors vary from red to blue as function values decrease from 1 to 0.
Spinor fermions: Zeeman term

- Even with optical trapping, external magnetic field $\mathcal{H}$ required to generate p-wave Feshbach resonance adds Zeeman term
  
  $\hat{H}_{\text{Zeeman}} = -g\mu\mathcal{H}\hat{S}_z$.

  Hamiltonian invariant under rotations in spin space $\Rightarrow$ Zeeman term shifts ground state energy without affecting wavefunction.
Spinor fermions: Zeeman term

- Even with optical trapping, external magnetic field $\mathcal{H}$ required to generate p-wave Feshbach resonance adds Zeeman term $\hat{H}_{\text{Zeeman}} = -g\mu_B \hat{S}_z$.
  
  Hamiltonian invariant under rotations in spin space $\Rightarrow$ Zeeman term shifts ground state energy without affecting wavefunction.

- Magnetization and total spin $S$ can be changed by a microwave pulse, putting system into a metastable excited state. Lifetime limited by dipolar interaction, negligible over experimental lifetimes for $^{40}\text{K}$: C. Ticknor, C.A. Regal, D.S. Jin, and J.L. Bohn, Phys. Rev. A 69, 042712 (2004).
Spinor fermions: Zeeman term

- Even with optical trapping, external magnetic field $\mathcal{H}$ required to generate p-wave Feshbach resonance adds Zeeman term $\hat{H}_{\text{Zeeman}} = -g\mu_\text{B}\hat{S}_z$.

Hamiltonian invariant under rotations in spin space $\Rightarrow$ Zeeman term shifts ground state energy without affecting wavefunction.

- Magnetization and total spin $S$ can be changed by a microwave pulse, putting system into a metastable excited state. Lifetime limited by dipolar interaction, negligible over experimental lifetimes for $^{40}\text{K}$: C. Ticknor, C.A. Regal, D.S. Jin, and J.L. Bohn, Phys. Rev. A 69, 042712 (2004).

- $g_2$ and $g_3$ in these states vs. $\sigma = S/N$ can be calculated by same variational method as before:
Spinor fermions, $g_2$ and $g_3$ vs. $\sigma$

$g_2(\gamma, \gamma')/n^2$
$g_2(\gamma, \sigma; \gamma')/n^3$ and $g_3(\gamma, \sigma; \gamma')/n^3$ in the $\gamma,\sigma$-plane, for $\gamma' = 0.0124$ as in Ticknor et al. experiment.